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THEORY OF THE TORSION BALANCE

**WITH A PRELIMINARY STUDY
OF A MODIFICATION OF THE INSTRUMENT
TO DECREASE TIME OF GRAVITY
MEASUREMENTS**

BY

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THEORY OF THE TORSION BALANCE; WITH A PRELIMINARY STUDY OF A MODIFICATION OF THE INSTRUMENT TO DECREASE TIME OF GRAVITY MEASUREMENTS¹

By J. WALLACE JOYCE²

INTRODUCTION

Within the past few years geophysical methods of prospecting have achieved singular success in the location of oil-producing salt domes in southeast Texas and Louisiana. The Eötvös torsion balance has played an indispensable rôle in this work and is largely responsible for the discovery and delineation of numerous domes. It has also been applied with satisfactory results to the location of buried ridges, gorges, faults, large ore deposits, and similar formations and structures in regions where the geology and topography of the country permit its use.

Those engaged in torsion-balance operation are continually striving to improve the instrument. The two problems now receiving greatest attention are improvement of suspension fibers and reduction of the time required to make observations, that is, reduction in the oscillation period of the suspended system of the balance.

To produce a torsion balance capable of yielding consistent readings a suspension fiber possessing invariable physical properties under stress must be used. A section of this report is devoted to certain broader aspects of the preparation of fibers for use in torsion balances.

Reducing the free period of oscillation of the suspended system of the torsion balance produces a corresponding reduction in the time required to make observations. The advantages of such a change from the operator's viewpoint are obvious; the speed of operation of the balance is increased, and the disturbing influences of temperature variations are reduced.

This paper has been prepared with several definite purposes: (1) To present a theoretical discussion of the principles and properties of the gravity field and the interrelation of the physical quantities involved to the torsion-balance equation; (2) to consider some of the practical features of the torsion balance; and (3) to describe an investigation of the problem of reducing the free period of the balance system. Modification of the present torsion balance, to reduce materially the free period of oscillation without sacrifice of sensitivity, was also investigated. Such modification implies a stiffer suspension and the development of a method capable of measuring extremely minute angular deflections.

As a further refinement F. W. Lee, of the Bureau of Mines, proposes equipment of the balance with an extremely fine restoring

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torsion fiber so that in operating it the suspended system can always be returned to an arbitrarily chosen initial position. The twist of this fiber is then a measure of the force acting on the suspended system. With this scheme of operation a means of measuring small angular displacements need not be devised—simply a method of detecting them so that the balance arm can always be returned to its initial position.

To the knowledge of the author no American scientific-instrument manufacturer has designed and built a torsion balance for gravity surveys. He hopes that this publication may stimulate enough interest in prospective manufacturers to prompt them to make a rapid, precise instrument for gravity surveying.

ACKNOWLEDGMENTS

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GRAVITY FIELD

There are two ways or methods of analyzing gravity fields of force:

(1) By determining the force from every portion of a mass and then combining vectorially all component forces into a final resultant force.

(2) By determining the work or potential of a field, thereby fixing a system of equipotential surfaces. This involves, first, integration or summation of the work of the total mass upon a unit mass. The forces may now be determined by differentiation of this potential function in the various directions, giving the potential gradient or force. If, for example, the differentiation is taken in the three cardinal directions the separate components can be easily combined into a single resultant.

In this discussion the latter method has been found more convenient and will be used.

The results of a torsion-balance survey are based on interpretation of local anomalies produced on the normal gravity field of the earth by buried bodies or structures. The Eötvös balance measures two of these anomalies—the horizontal gradient of gravity and a curvature function of the equipotential surface.

The horizontal gradient of gravity means the change in the vertical force of gravity per unit horizontal displacement.

The curvature function is a measure of the distortion of the surfaces of equal gravity potential from the spherical forms they would theoretically assume if the earth were a homogeneous sphere at rest. The minimum and maximum radii of curvature of the portion of the actual equipotential surface within the balance system are denoted by ρ_1 and ρ_2 , respectively. If g is the total force of gravity at the center of mass of the balance system the curvature function used is defined as follows:

$$\text{Curvature function} = g \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right).$$

Thus, for a spherical surface where all radii of curvature are equal ($\rho_1 = \rho_2$) the curvature function becomes zero. For a distorted surface $\rho_1 \neq \rho_2$ and $g\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$ assumes a value the magnitude of which increases with the degree of distortion.

The object of this discussion of the gravity field is to determine the gradient and curvature function in terms of a series of intermediate quantities which may be associated with the equation for operation of the torsion balance. Thus observations of the balance will enable the investigator to compute these intermediate quantities, and from them the gradient and curvature function can be determined.

GRAVITY POTENTIAL

The starting point in the development of a mathematical study of the gravity field is the definition of potential. At every point in the gravity field there exists a gravitational potential, defined as the work done when unit mass moves from infinity to the point at which the potential is to be determined. It follows that the potential difference between two points at a finite distance apart represents the work done when unit mass moves from one point to the other.

The gravitational force between two bodies varies as the inverse square of the distance between their centers of mass. Therefore, the potential can be computed at any point, P , in space, due to a mass concentrated at Q at a distance, r , from P . (See fig. 1, A .)

Let

m = mass of object at Q , and

G = proportionality constant known as universal gravitational constant = 6.66×10^{-8} , using C. G. S. units.

Then the force on unit mass at distance r from Q will be

$$f = G \frac{m \times 1}{r^2}. \quad (1)$$

The work done in moving unit mass through a distance, Δr , then becomes

$$\Delta w = f \Delta r = G \frac{m}{r^2} \Delta r. \quad (2)$$

The total work in moving unit mass from infinity to point P —that is, the potential of the gravity field at P due to mass m at Q —is the integral of Δw from infinity to r_1 :

$$V_P = \int_{\infty}^{r_1} \Delta w = \int_{\infty}^{r_1} G \frac{m}{r^2} dr = -\frac{Gm}{r_1}. \quad (3)$$

The negative sign indicates that the force of gravity is an attraction; therefore the work is done by unit mass in moving from infinity to P .

So far it has been assumed that the mass of the attracting body can be concentrated at a point, that is, that the physical dimensions of the body are infinitely small. Now consider mass M , which can not be so concentrated. Here the body is divided into elementary masses Δm , and the potential at P (fig. 1, B) due to each of these elements is determined. The total potential at P due to M is the

sum of all the elementary potentials as the elementary masses are made increasingly smaller. Expressed as an equation

$$V_P = \lim_{\Delta m \rightarrow 0} G \sum \frac{\Delta m}{r} \quad (4)$$

r = mean distance of each element, Δm , from P , the point at which the potential is to be evaluated.

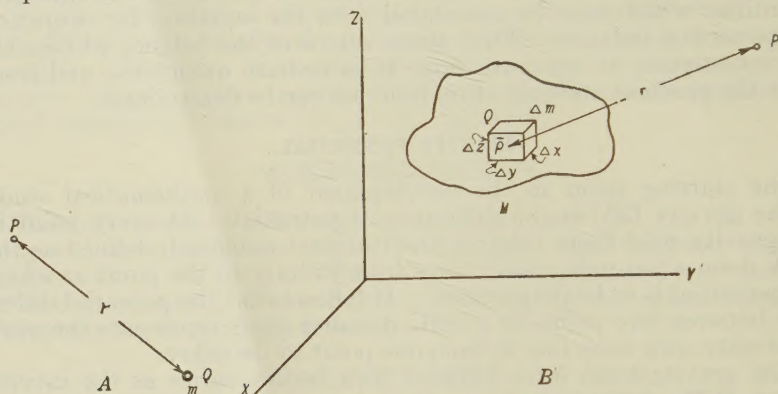


FIGURE 1.—A, Basis for computation of gravity potential due to a mass point; B, basis for computation of gravity potential due to a mass of finite dimensions

The mass Δm may be expressed in terms of an elementary volume times an average density. Thus, let

$\bar{\rho}$ = average density of the element.

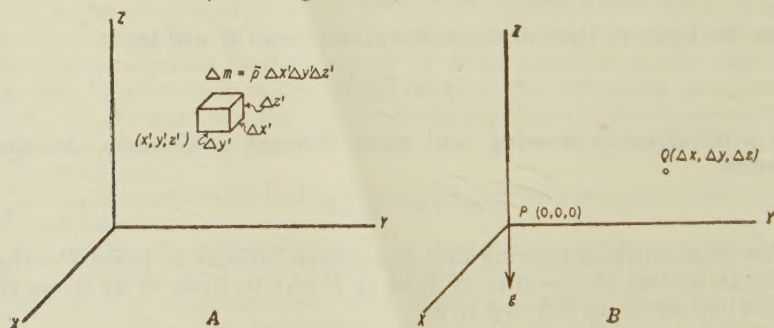


FIGURE 2.—A, Elemental mass; B, gravitational potential in space

Then, if x' , y' , and z' are the coordinates of that corner of Δm nearest the origin, as shown in Figure 2, A, and $\Delta x'$, $\Delta y'$, and $\Delta z'$ are the linear dimensions of the element,

$$\Delta m = \bar{\rho} \Delta x' \Delta y' \Delta z' \quad (5)$$

Replacing the summation in (4) by a triple integration covering the entire body M , the potential becomes

$$\left. \begin{aligned} V_P &= G \iiint \frac{\bar{\rho} dx' dy' dz'}{r} \\ \text{or,} \quad V_P &= G \iiint \frac{\bar{\rho} dx' dy' dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} \end{aligned} \right\} \quad (6)$$

Thus it is evident that the value of gravitational potential V_P is a function of the three space coordinates and, in general,

$$V_o = f(x, y, z). \quad (7)$$

The exact form of function $f(x, y, z)$ depends on the physical conditions of the problem, such as the shape of the attracting object, its distance from P , and the manner in which its density is distributed.

The first elementary theorem regarding fields states that the force associated with the field is proportional to the potential gradient or the rapidity with which work is done in moving in the direction of the field. To simplify the mathematics it is easier to determine the gradient in the three directions at right angles to each other and then combine them into one resultant.

Hence, if V is the potential of the gravity field the force of gravity in a given direction is obtained by differentiating V in that direction. Thus, the three components of gravity become

$$\left. \begin{aligned} g_x &= \frac{\partial V}{\partial x} \\ g_y &= \frac{\partial V}{\partial y} \\ g_z &= \frac{\partial V}{\partial z} \end{aligned} \right\} \quad (8)$$

The directions of x, y , and z are mutually at right angles to each other.

The total force of gravity is the vector sum of these three components; therefore

$$g = \sqrt{(g_x)^2 + (g_y)^2 + (g_z)^2}. \quad (9)$$

A further differentiation of a given component in a given direction determines the gradient or rate of change of the component in this direction. Thus, for the X component:

$$\left. \begin{aligned} \frac{\partial g_x}{\partial x} &= \frac{\partial^2 V}{\partial x^2} = V_{xx} = \text{gradient of } g_x \text{ in } X \text{ direction} \\ \frac{\partial g_x}{\partial y} &= \frac{\partial^2 V}{\partial x \partial y} = V_{xy} = \text{gradient of } g_x \text{ in } Y \text{ direction} \\ \frac{\partial g_x}{\partial z} &= \frac{\partial^2 V}{\partial x \partial z} = V_{xz} = \text{gradient of } g_x \text{ in } Z \text{ direction} \end{aligned} \right\} \quad (10a)$$

For the Y component:

$$\left. \begin{aligned} \frac{\partial g_y}{\partial x} &= \frac{\partial^2 V}{\partial y \partial x} = V_{yx} = \text{gradient of } g_y \text{ in } X \text{ direction} \\ \frac{\partial g_y}{\partial y} &= \frac{\partial^2 V}{\partial y^2} = V_{yy} = \text{gradient of } g_y \text{ in } Y \text{ direction} \\ \frac{\partial g_y}{\partial z} &= \frac{\partial^2 V}{\partial y \partial z} = V_{yz} = \text{gradient of } g_y \text{ in } Z \text{ direction} \end{aligned} \right\} \quad (10b)$$

A similar set of equations may also be written for the Z component.

HORIZONTAL GRADIENT OF GRAVITY

Although, intrinsically, little is known of the exact nature of fields, much work has been done in describing them. Here it is necessary to point out two general types of fields: (1) Those that, if followed, close

upon themselves and (2) those that extend to infinity and do not close upon themselves. The first are rotational fields (a magnetic field is the best example); the second are irrotational fields (the gravity field is a good illustration).

Both fields include nine gradients, as each of the three components has a gradient corresponding to each of the three coordinate directions in space. However, since the gravity field is an irrotational vector field, several of these quantities are equal, so that in reality there are but six independent gradients. The equalities, which follow, are necessary to define a field of this nature:

$$\left. \begin{aligned} V_{xy} &= V_{yx} \\ V_{xz} &= V_{zx} \\ V_{yz} &= V_{zy} \end{aligned} \right\}. \quad (11)$$

At the origin and within the balance system itself, since $g_z \gg g_x$ or g_y , for all practical considerations $g_z = g$; that is, the Z or vertical component of gravity is essentially equal to the total force of gravity. Therefore, equations (10) and (11) may be written:

$$\left. \begin{aligned} \frac{\partial g_z}{\partial x} &= V_{xz} = V_{zx} = \frac{\partial g}{\partial x} = \text{gradient of gravity in } X \text{ direction} \\ \frac{\partial g_z}{\partial y} &= V_{yz} = V_{zy} = \frac{\partial g}{\partial y} = \text{gradient of gravity in } Y \text{ direction} \end{aligned} \right\}. \quad (12)$$

From these components may be formed the "horizontal gradient of gravity," that is, the change in the vertical force of gravity for unit horizontal displacement.

As V_{xz} and V_{zy} represent the change of the vertical component of gravity in the directions of X and Y , respectively:

$$\begin{aligned} \text{Horizontal gradient of the vertical gravity component} \\ = Gr_g = \frac{\partial g}{\partial s} = \sqrt{V_{xz}^2 + V_{zy}^2}. \end{aligned} \quad (13)$$

This is generally called the horizontal gradient of gravity.

The horizontal gradient makes angle θ_1 with the positive X axis, where

$$\theta_1 = \tan^{-1} \frac{V_{zy}}{V_{xz}}. \quad (14)$$

It follows that since θ_1 gives the direction of the maximum horizontal gradient the direction of lines of constant gravity or isogams will be at right angles to this direction, that is, at angle θ_2 with the positive X axis, where

$$\theta_2 = \tan^{-1} - \frac{V_{xz}}{V_{zy}}. \quad (15)$$

GRAVITY-EQUIPOTENTIAL SURFACES

As the masses on the suspension system of the balance are separated by a small distance and the instrument deflection depends on the difference in gravity between these two points, it is essential to consider the value of gravity at adjacent point Q . Points P and Q , as will be shown later, bear directly upon the evaluation of the gravity field by the instrument.

Now the value of gravity at adjacent or near-by point Q does not differ greatly from the value of P . It may be expressed as a correction, using the usual methods of correction coefficients α and β as a function of r , the distance between points P and Q . Thus,

$$V_Q = V_P + \alpha r + \beta r^2 + \text{terms of higher order.}$$

Now α and β differ for every value of r as well as every direction of r from P ; a stronger mathematical tool must therefore be used to eliminate the infinite numbers of α 's and β 's. Taylor's theorem combines all the different values of α and β into one function of surfaces passing through points P and Q .

To study the form of the surfaces of equal gravity potential the idea of potential must be extended to a set of points using Taylor's theorem.

Let

V_P = potential of gravity at $P(x, y, z)$,

V_Q = potential of gravity at any adjacent point, $Q(x + \Delta x, y + \Delta y, z + \Delta z)$.

Then

$$V_P = f(x, y, z). \quad (7)$$

In like manner,

$$V_Q = f(x + \Delta x, y + \Delta y, z + \Delta z). \quad (16)$$

Expanding (16) by Taylor's series:

$$\begin{aligned} V_Q = V_P + \Delta x \frac{\partial V}{\partial x} + \Delta y \frac{\partial V}{\partial y} + \Delta z \frac{\partial V}{\partial z} + \frac{1}{2} \left[\overline{\Delta x^2} \frac{\partial^2 V}{\partial x^2} + \overline{\Delta y^2} \frac{\partial^2 V}{\partial y^2} + \overline{\Delta z^2} \frac{\partial^2 V}{\partial z^2} \right. \\ \left. + 2\Delta x \Delta y \frac{\partial^2 V}{\partial x \partial y} + 2\Delta x \Delta z \frac{\partial^2 V}{\partial x \partial z} + 2\Delta y \Delta z \frac{\partial^2 V}{\partial y \partial z} \right] + \epsilon, \end{aligned} \quad (17)$$

where ϵ = sum of terms of higher order.

The addition of higher-order terms increases the accuracy of the approximation of V_Q to its true value, but for all practical purposes terms of higher order than the second need not be considered; consequently, the omission of ϵ will not cause appreciable error in the computations.

For convenience, let the origin of a Cartesian system of coordinates coincide with point P and let the axes be so oriented that at the origin the total force of gravity parallels the Z axis. (See fig. 2, B.) Then, for points in the vicinity of the origin the horizontal components of gravity will be negligible, and the vertical component will approximate closely the total force. Thus,

$$\left. \begin{aligned} \frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = 0 \\ \frac{\partial V}{\partial z} = g = \text{total force of gravity} \end{aligned} \right\} \text{at and near the origin.} \quad (18)$$

Equation (17) then reduces to:

$$\begin{aligned} V_Q = V_P + g \Delta z + \frac{1}{2} \left[\overline{\Delta x^2} \frac{\partial^2 V}{\partial x^2} + \overline{\Delta y^2} \frac{\partial^2 V}{\partial y^2} + \overline{\Delta z^2} \frac{\partial^2 V}{\partial z^2} + 2\Delta x \Delta y \frac{\partial^2 V}{\partial x \partial y} \right. \\ \left. + 2\Delta x \Delta z \frac{\partial^2 V}{\partial x \partial z} + 2\Delta y \Delta z \frac{\partial^2 V}{\partial y \partial z} \right]. \end{aligned} \quad (19)$$

The dimensions of the torsion-balance system are vanishingly small compared to those of the equipotential surfaces of the gravity field. Consequently, within the balance system

$$x = \Delta x, y = \Delta y, z = \Delta z. \quad (20)$$

Also, all changes of the gravity field within the balance system can be taken as uniform; that is, the equipotential surfaces are assumed to be quadric surfaces with no discontinuities, an assumption readily justified by actual experience. As the result of this last condition all second partial derivatives become constants; they are usually designated as follows:

$$\left. \begin{aligned} \frac{\partial^2 V}{\partial x^2} &= V_{xx}, \frac{\partial^2 V}{\partial x \partial y} = V_{xy} \\ \frac{\partial^2 V}{\partial y^2} &= V_{yy}, \frac{\partial^2 V}{\partial x \partial z} = V_{xz} \\ \frac{\partial^2 V}{\partial z^2} &= V_{zz}, \frac{\partial^2 V}{\partial y \partial z} = V_{yz} \end{aligned} \right\} \quad (21)$$

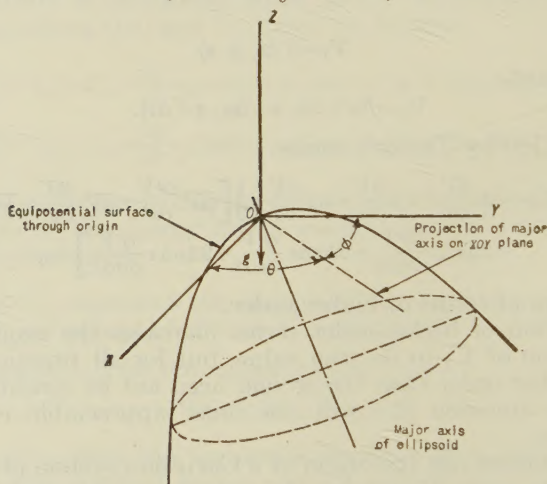


FIGURE 3.—Gravitational equipotential surface

Substituting these values, equation (19) may now be rewritten:

$$V_Q = V_P + gz + \frac{1}{2} \left[x^2 V_{xx} + y^2 V_{yy} + z^2 V_{zz} + 2xy V_{xy} + 2xz V_{xz} + 2yz V_{yz} \right]. \quad (22)$$

To determine the expression for an equipotential surface both potentials V_Q and V_P must be assumed to have the same value. Then

$$gz + \frac{1}{2} V_{xx} x^2 + \frac{1}{2} V_{yy} y^2 + \frac{1}{2} V_{zz} z^2 + V_{xy} xy + V_{xz} xz + V_{yz} yz = 0. \quad (23)$$

This is the equation of an ellipsoid passing through the origin with major axis inclined to the XOZ and YOZ planes, as shown in Figure 3. Further investigation of the equation will show that this axis makes an angle of θ with the XOZ plane and an angle of ϕ with the YOZ plane, where θ and ϕ are determined by the relations:

$$\tan 2\theta = - \frac{2V_{xz}}{V_{zz} - V_{xx}}. \quad (24)$$

$$\tan 2\phi = - \frac{2V_{yz}}{V_{zz} - V_{yy}}. \quad (25)$$

CURVATURE FUNCTION

As already pointed out, the level surfaces of gravity for a homogeneous spheroid would assume spheroidal configurations. However, in the presence of a heterogeneous structure, such as the earth's crust, these level surfaces are subjected to local distortions. To determine the extent of these distortions the radii of curvature of the equipotential surfaces must be ascertained in two planes perpendicular to each other, whose intersection lies along the Z axis. These planes are chosen so that one cuts the surface along the direction of maximum radius of curvature and the other in the direction of the minimum radius of curvature.

To compute these various quantities pass a plane through the origin perpendicular to the XOY plane, making an angle, ψ , with the XOZ

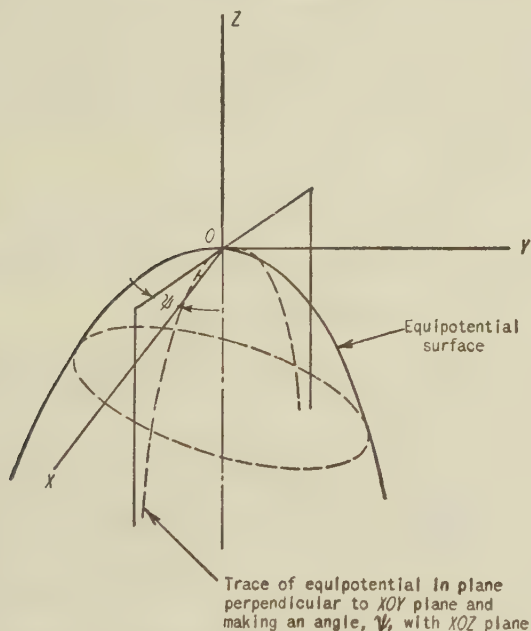


FIGURE 4.—Determination of direction of maximum and minimum radii of curvature of equipotential surface

plane, as shown in Figure 4. Introducing cylindrical coordinates, the trace of the equipotential surface in this plane is found by substituting in equation (23) the quantities, $x=r \cos \psi$, $y=r \sin \psi$, $z=z$:

$$gz + \frac{1}{2} V_{xx} r^2 \cos^2 \psi + \frac{1}{2} V_{yy} r^2 \sin^2 \psi + \frac{1}{2} V_{zz} z^2 + V_{xy} r^2 \sin \psi \cos \psi + V_{xz} r z \cos \psi + V_{yz} r z \sin \psi = 0. \quad (26)$$

The radius of curvature is calculated from the usual formula:

$$\rho = \frac{\left[1 + \left(\frac{dz}{dr} \right)^2 \right]^{\frac{3}{2}}}{\frac{d^2 z}{dr^2}}. \quad (27)$$

Quantities $\frac{dz}{dr}$ and $\frac{d^2 z}{dr^2}$ are obtained by differentiating equation (26).

As the balance system is located at the origin, the curvature at that point is required and is formed by letting $z=r=0$. The radius of curvature ρ is thus determined to be,

$$\rho = \frac{-g}{V_{xx} \cos^2 \psi + V_{yy} \sin^2 \psi + 2V_{xy} \sin \psi \cos \psi} \quad (28)$$

To determine the value of angle ψ which makes ρ a maximum or a minimum it is necessary to take the first derivative of (28) with respect to ψ , equate it to zero, and solve for ψ .

$$\left. \begin{aligned} \frac{d\rho}{d\psi} &= \frac{-2g[-V_{xx} \sin \psi \cos \psi + V_{yy} \sin \psi \cos \psi + V_{xy}(\cos^2 \psi - \sin^2 \psi)]}{[V_{xx} \cos^2 \psi + V_{yy} \sin^2 \psi + 2V_{xy} \sin \psi \cos \psi]^2} \\ \tan 2\psi &= -\frac{2V_{xy}}{V_{yy} - V_{xx}} \end{aligned} \right\} \quad (29)$$

As $\tan 2\psi$ and $\tan \psi$ bear a quadratic relation to each other equation (29) will determine two values of ψ , one of which corresponds to maximum curvature and the other minimum curvature. Let ψ_1 and ψ_2 , respectively, denote these two angles. As the equipotential surface is an ellipsoid, the directions of maximum and minimum curvature lie $\frac{\pi}{2}$ apart. Hence,

$$\left. \begin{aligned} \cos \psi_1 &= -\sin \psi_2 \\ \sin \psi_1 &= \cos \psi_2 \end{aligned} \right\} \quad (30)$$

As already stated, the curvature function may be defined as:

$$\text{Curvature function} = g \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \quad (31)$$

To avoid confusion, ρ_1 and ρ_2 are chosen so that this quantity will always be positive, for in certain configurations one of the radii of curvature may be negative.

To evaluate the curvature function in terms of the gravity-field constants the reciprocals of maximum and minimum radii of curvature are determined, using ψ_1 and ψ_2 , the two angles determined by equation (29). Equation (30) is then employed to express the results in terms of ψ_1 . This gives

$$\left. \begin{aligned} g \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) &= (V_{yy} - V_{xx})(\cos^2 \psi_1 - \sin^2 \psi_1) - 4V_{xy} \sin \psi_1 \cos \psi_1 \\ \text{or} \quad g \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) &= (V_{yy} - V_{xx}) \cos 2\psi_1 - 2V_{xy} \sin 2\psi_1 \end{aligned} \right\} \quad (32)$$

Multiplying both sides of (32) by $\frac{\sin 2\psi_1}{2}$,

$$\frac{g}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \sin 2\psi_1 = \frac{(V_{yy} - V_{xx})}{2} \cos 2\psi_1 \sin 2\psi_1 - V_{xy} \sin^2 2\psi_1 \quad (33)$$

But from (29)

$$\frac{V_{yy} - V_{xx}}{2} = -\frac{V_{xy}}{\tan 2\psi_1} = -\frac{V_{xy} \cos 2\psi_1}{\sin 2\psi_1} \quad (34)$$

Therefore

$$g \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \sin 2\psi_1 = -2V_{xy} (\cos^2 2\psi_1 + \sin^2 2\psi_1) = -2V_{xy} \quad (35)$$

Returning to (29),

$$V_{xy} = -\frac{1}{2}(V_{yy} - V_{xx}) \tan \psi_1.$$

Using this relation in conjunction with equation (35) angle ψ_1 may be eliminated, giving as the final expression for the curvature function

$$g\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = \sqrt{(V_{yy} - V_{xx})^2 + 4V_{xy}^2}. \quad (36)$$

Equation (36) indicates that if $(V_{yy} - V_{xx})$ and V_{xy} are known function $g\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right)$ may always be found. This function will then give a measure of the maximum and minimum radii of curvature of the level surface, indicating its deviation from the ideal case of a spherical surface.

In the special case in which the coordinate axes coincide with the directions of maximum and minimum radii of curvature of the equipotential surface angle ψ_1 becomes equal to zero and from equation (29) $V_{xy} = 0$. Equation (36) then reduces to the form:

$$g\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = V_{yy} - V_{xx}. \quad (37)$$

For the convenience of the reader the various relations derived in this section on the gravity field are now summarized.

SUMMARY OF RELATIONS GIVING PROPERTIES OF GRAVITY FIELD

$V = f(x, y, z)$ = potential of gravity.

$$\left. \begin{aligned} \frac{\partial V}{\partial x} &= g_x \\ \frac{\partial V}{\partial y} &= g_y \\ \frac{\partial V}{\partial z} &= g_z \end{aligned} \right\} \text{Components of gravity, mutually perpendicular to each other.}$$

$g = \sqrt{(g_x)^2 + (g_y)^2 + (g_z)^2}$, total force of gravity.

$$\left. \begin{aligned} \frac{\partial^2 V}{\partial x^2} &= V_{xx} \\ \frac{\partial^2 V}{\partial x \partial y} &= V_{xy} \\ \frac{\partial^2 V}{\partial x \partial z} &= V_{xz} \end{aligned} \right\} \begin{aligned} &\text{Gradients of} \\ &X \text{ component} \\ &\text{of gravity.} \end{aligned} \quad \left. \begin{aligned} \frac{\partial^2 V}{\partial y \partial x} &= V_{yx} \\ \frac{\partial^2 V}{\partial y^2} &= V_{yy} \\ \frac{\partial^2 V}{\partial y \partial z} &= V_{yz} \end{aligned} \right\} \begin{aligned} &\text{Gradients of} \\ &Y \text{ component} \\ &\text{of gravity.} \end{aligned}$$

$$\left. \begin{aligned} \frac{\partial^2 V}{\partial z \partial x} &= V_{zx} \\ \frac{\partial^2 V}{\partial z \partial y} &= V_{zy} \\ \frac{\partial^2 V}{\partial z^2} &= V_{zz} \end{aligned} \right\} \begin{aligned} &\text{Gradients of} \\ &Z \text{ component} \\ &\text{of gravity.} \end{aligned}$$

$$V_{xy} = V_{yx}, \quad V_{xz} = V_{zx}, \quad V_{yz} = V_{zy}.$$

To a close approximation:

$$\left. \begin{aligned} \frac{\partial g_x}{\partial x} &= V_{xx} = V_{zx} = \frac{\partial g}{\partial x} \\ \frac{\partial g_x}{\partial y} &= V_{xy} = V_{yz} = \frac{\partial g}{\partial y} \\ \frac{\partial g_x}{\partial z} &= V_{xz} = \frac{\partial g}{\partial z} \end{aligned} \right\} \begin{aligned} &\text{Gradients of } g, \text{ the} \\ &\text{total force of gravity.} \end{aligned}$$

$$Gr_o = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2} = \sqrt{(V_{xx})^2 + (V_{yy})^2}, \text{ horizontal gradient of gravity.}$$

$$q\left(\frac{1}{\rho_1} - \frac{1}{\rho_2}\right) = \sqrt{(V_{yy} - V_{xx})^2 + 4V_{xy}^2}, \text{ curvature function of gravity.}$$

TORSION BALANCE

THEORY AND DERIVATION OF TORSION-BALANCE EQUATION

The equation of operation of the torsion balance may be derived by graphical or analytical methods. The former method lends itself more readily to giving a physical picture of the forces involved and their action on various parts of the balance system. Accordingly, the graphical solution will be employed in this section of the paper, and for the benefit of readers who may be interested the analytical method is given in Appendix A.

The essential features of the Eötvös torsion balance are illustrated in Figure 5, *A*. They include:

- (a) A torsion wire suspended from a torsion head at *A*.
- (b) A light horizontal beam fastened to the lower end of the torsion wire at *B*, the balance point of the suspended system.
- (c) A small weight, m' , fastened to the beam at *C*.
- (d) An equal weight, m , suspended by a fine wire from the balance beam at *D*.

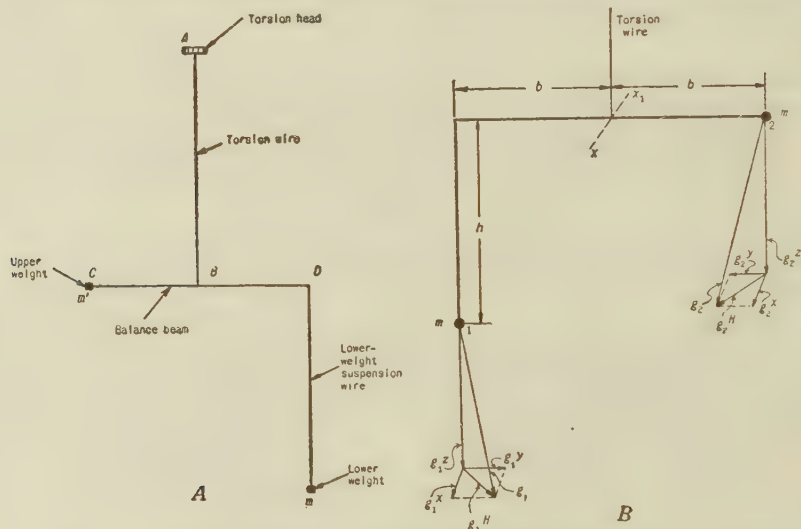


FIGURE 5.—*A*, elements of Eötvös torsion balance; *B*, forces on Eötvös torsion balance

When such a system is placed in a gravitational field the force of gravity acts on the various masses; and in general a turning moment or torque, which causes rotation of the suspended system about the axis of the torsion wire, will be produced.

The problem is to determine a relation between the angular displacement of the balance bar and the forces acting on the various masses of the system. The equations for these forces contain the quantities V_{xx} , V_{yy} , V_{xy} , V_{zx} , and V_{zy} appearing in the expressions for the horizontal gradient and curvature function of the gravity field. Consequently a relation between angular displacement and the second derivatives of gravity potential V_{xx} , V_{yy} , etc., can be indicated, permitting determination of the latter. These in turn are used to compute the horizontal gradient and curvature function.

The forces acting on the balance system are illustrated in Figure 5, *B*. The total forces of gravity acting on the weights at 1 and 2 are represented by vectors g_1 and g_2 , respectively. These forces are resolved into their three components: g_1^z and g_2^z , the vertical, and g_1^x, g_2^x, g_1^y and g_2^y , the horizontal forces. The total horizontal forces are denoted by g_1^H and g_2^H , respectively.

The normal vertical gradient of gravity due to changes in distance from the center of mass of the earth is approximately 3.1×10^3 Eötvös units at mean sea level. As a result of this normal gradient the force

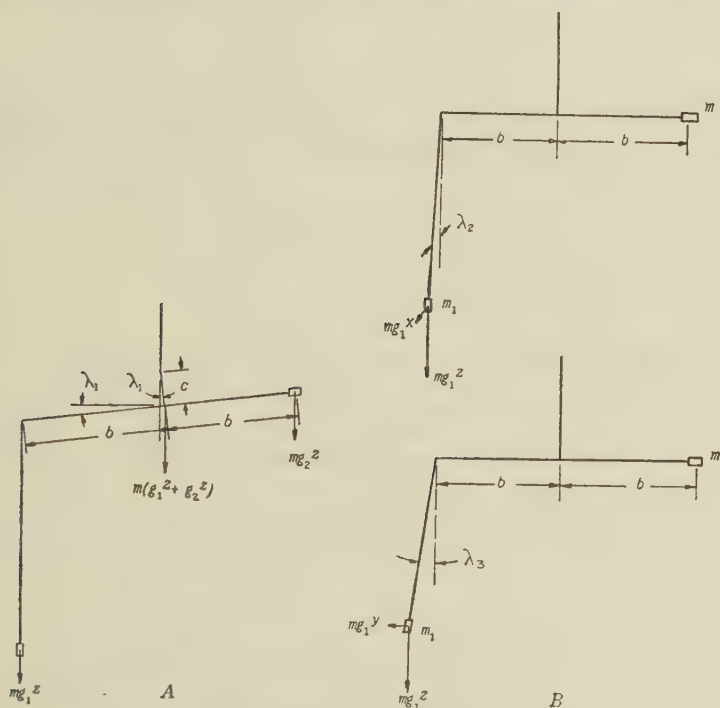


FIGURE 6.—*A*, effect of vertical component of gravity on torsion-balance system; *B*, effect of horizontal forces on lower-weight suspension

of gravity at the lower weight is greater than that at the upper weight, causing slight tipping of the balance beam about a horizontal axis. From the balance dimensions a close approximation of λ_1 , the angle of tipping, may be found as follows:

At equilibrium, as seen in Figure 6, *A*,

$$m(g_1^z - g_2^z)b \cos \lambda_1 = m(g_1^x + g_2^x)c \sin \lambda_1,$$

$$\lambda_1 = \tan^{-1} \frac{g_1^z - g_2^z}{g_1^x + g_2^x} \frac{b}{c},$$

$$\frac{g_1^z - g_2^z}{g_1^x + g_2^x} = \frac{(3.1 \times 10^{-6})h}{1.96 \times 10^3} = 1.58h \times 10^{-9}.$$

As a maximum for most balances $\frac{hb}{c} = 200$.

Then

$$\lambda_1 = \tan^{-1} 3.16 \times 10^{-7},$$

or

$$\lambda_1 = 0.06 \text{ second of arc.}$$

This angle is perpendicular to the angle of torque and is negligible in all practical cases.

A similar analysis may be applied to the angle that the lower-weight suspension wire makes with the vertical as a result of the maximum horizontal component of gravity encountered. Calling the vertical angles in the X and Y directions λ_2 and λ_3 , respectively (fig. 6, B),

$$\lambda_2 = \tan^{-1} \frac{g_1^x}{g_1^z}, \quad \lambda_3 = \tan^{-1} \frac{g_1^y}{g_1^z}.$$

Taking as a maximum value for the gradient of the horizontal intensity 30 E, ratios $\frac{g_1^x}{g_1^z}$ and $\frac{g_1^y}{g_1^z}$ become equal to approximately 6.1×10^{-10}

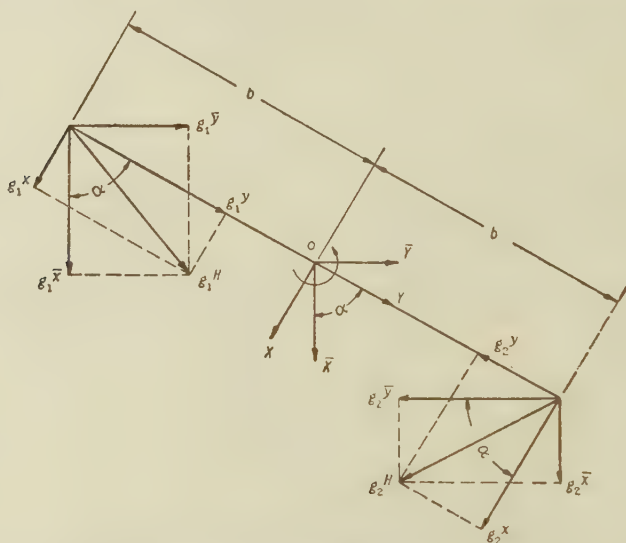


FIGURE 7.—Horizontal forces on torsion-balance system

for a balance in which $b = 20$ cm. Thus, λ_2 and λ_3 are approximately 1.2×10^{-4} second of arc and are therefore negligible.

As a result of these considerations the only appreciable displacement of the torsion balance is a rotation of the suspended system about the axis of the torsion wire; furthermore this rotation is due entirely to the difference in horizontal components of the force of gravity illustrated in Figure 7.

In Figure 5, B , the horizontal force of gravity was resolved into components parallel and perpendicular to the balance arm. A new set of coordinate axes would therefore be required for each new position of the balance. This restriction is overcome by introducing a new set of axes, $O\bar{X}$ and $O\bar{Y}$, making an angle $\left(\frac{\pi}{2} - \alpha\right)$ with the original set, as shown in Figure 7. Angle α is usually measured from

a north-south line to the balance beam, and its magnitude is at the disposal of the operator.

Taking counterclockwise rotation as positive, the initial step in the solution of the problem is to write the moments of the forces about the vertical axis of rotation at O .

For the convenience of the reader the various symbols used throughout this paper in connection with the torsion balance are listed below.

L =length of torsion wire, centimeters.

a =diameter of torsion wire, centimeters.

μ =coefficient of rigidity of torsion-wire material, C. G. S. units.

τ =rigidity constant of torsion wire, C. G. S. units= $\frac{\pi a^4}{2L}\mu$.

b =one-half length of balance arm, centimeters.

c =length of rigid piece connecting balance beam to torsion wire, centimeters.

m =mass of each weight, grams.

h =length of suspension of the lower weight, centimeters.

θ =angular position of balance beam as read on scale, radians.

θ_0 =zero position, radians.

α =azimuth of balance system, radians, that is, angle between balance arm and reference axis, usually north-south line.

The moment (M) of the horizontal forces about O , shown in Figure 7, is

$$M = (g_1^x - g_2^x)mb \sin \alpha - (g_1^y - g_2^y)mb \cos \alpha. \quad (38)$$

At equilibrium this moment is just equaled by T , the torque of the suspension, where

$$T = (\theta - \theta_0)\tau. \quad (39)$$

Therefore:

$$(\theta - \theta_0)\tau = (g_1^x - g_2^x)mb \sin \alpha - (g_1^y - g_2^y)mb \cos \alpha. \quad (40)$$

Equation (40) gives a relation between angular displacement and force. The force must now be determined in terms of the second derivatives V_{xy} , V_{xz} , etc. By definition,

$V_{xx} = \frac{\partial g_x}{\partial x}$ =rate of change of the X component of gravity in the X direction.

$V_{xy} = \frac{\partial g_x}{\partial y}$ =rate of change of the X component of gravity in the Y direction.

$V_{xz} = \frac{\partial g_x}{\partial z}$ =rate of change of the X component of gravity in the Z direction.

Similarly, the other second derivatives may be defined.

As already pointed out, these quantities are taken as constants within the torsion-balance system. Thus, in moving from the points 1 to 2 (fig. 5, B) the change in any particular component of g is computed by multiplying the distance moved in a given direction by the constant rate of change of that particular component. For example, in going from 1 to 2 the three components of motion are found to be:

Motion in X direction= $2b \cos \alpha$.

Motion in Y direction= $2b \sin \alpha$.

Motion in Z direction= h .

Change in X component of gravity in going from 1 to 2 is:

$$\Delta g^x = V_{xx}(2b \cos \alpha) + V_{xy}(2b \sin \alpha) + V_{xz}h. \quad (41)$$

Then

$$\begin{aligned} g_x^{\bar{x}} &= g_1^{\bar{x}} + \Delta g^{\bar{x}} \\ &= g_1^{\bar{x}} + 2V_{xx}b \cos \alpha + 2V_{xy}b \sin \alpha + V_{xz}h, \end{aligned}$$

and

$$(g_1^{\bar{x}} - g_x^{\bar{x}}) = -(2V_{xx}b \cos \alpha + 2V_{xy}b \sin \alpha + V_{xz}h). \quad (42)$$

Similarly,

$$\begin{aligned} g_x^{\bar{y}} &= g_1^{\bar{y}} + \Delta g^{\bar{y}} \\ &= g_1^{\bar{y}} + 2V_{yx}b \cos \alpha + 2V_{yy}b \sin \alpha + V_{yz}h, \end{aligned}$$

and

$$(g_1^{\bar{y}} - g_x^{\bar{y}}) = -(2V_{yx}b \cos \alpha + 2V_{yy}b \sin \alpha + V_{yz}h). \quad (43)$$

Substituting (42) and (43) in (40):

$$\begin{aligned} (\theta - \theta_0)\tau &= -(2V_{xx}b \cos \alpha + 2V_{xy}b \sin \alpha + V_{xz}h)mb \sin \alpha \\ &\quad + (2V_{yx}b \cos \alpha + 2V_{yy}b \sin \alpha + V_{yz}h)mb \cos \alpha. \end{aligned} \quad (44)$$

Since the gravity field is a vector field it can be shown that

$$\begin{aligned} V_{xy} &= V_{yx}, \\ V_{xz} &= V_{zx}, \\ V_{yz} &= V_{zy}. \end{aligned}$$

Substituting these relations in (44) and collecting terms:

$$(\theta - \theta_0)\tau = 2mb^2 \left[\frac{V_{yy} - V_{xx}}{2} \sin 2\alpha + V_{xy} \cos 2\alpha + \frac{h}{2b} V_{zy} \cos \alpha - \frac{h}{2b} V_{xz} \sin \alpha \right]. \quad (45)$$

The quantity $2mb^2$ is the moment of inertia of the two masses 1 and 2 (fig. 5, *B*) about the axis of the torsion wire. Theoretically, the moment of inertia of the balance beam and the suspension wire of the lower weight should also be considered. The masses of these two objects, however, are rendered negligible, as the beam is made of aluminum and the wire is kept extremely fine. As a result, no appreciable error is introduced by neglecting them. It may be pointed out that the analytical solution given in Appendix A takes account of these quantities.

Designating $2mb^2$ by the symbol K , equation (45) may be written:

$$(\theta - \theta_0)\tau = K \left(\frac{V_{yy} - V_{xx}}{2} \right) \sin 2\alpha + K V_{xy} \cos 2\alpha + mbh (V_{zy} \cos \alpha - V_{xz} \sin \alpha). \quad (46)$$

This is the torsion-balance equation.

In another section of this paper the statement is made that the Coulomb balance, in which the two masses are in the same horizontal plane, can measure the curvature function but not the gradient. This may be readily verified here. Equations (13) and (36) show that the curvature function involves quantities $(V_{yy} - V_{xx})$ and V_{xy} , whereas the gradient involves V_{xz} and V_{zy} . If both weights are in the same horizontal plane, as Figure 8, *A*, shows, terms V_{xz} and V_{zy} do not appear in the expressions for $g_x^{\bar{x}}$ and $g_x^{\bar{y}}$, since $h=0$. Consequently it is impossible to determine V_{xz} and V_{zy} , and the gradient therefore can not be measured with the Coulomb instrument.

Equation (46) shows that five unknown quantities are involved, $(V_{yy} - V_{xx})$, V_{xy} , V_{xz} , V_{zy} , and θ_0 , the latter being the position of the balance if no gravity field were present. Angle θ is observed, and quantities K , m , b , and h are determined by the physical dimensions and masses of the instrument. Angle α is left to the choice of the operator. To determine the five unknowns five equations, involving five known values of α , are requisite. Physically this means that at a given station five readings of the balance are taken, corresponding to

five different orientations of the balance system with a given reference axis, usually the north-south meridian.

SOLUTION OF TORSION-BALANCE EQUATION

In solving equation (46) for the five unknowns—($V_{yy} - V_{xx}$), V_{xy} , V_{zz} , V_{zy} , and θ_0 —five readings of the balance corresponding to five arbitrary values of α are necessary. Actually, calculations are simplified by using six values of α , namely, 0° , 60° , 120° , 180° , 240° ,

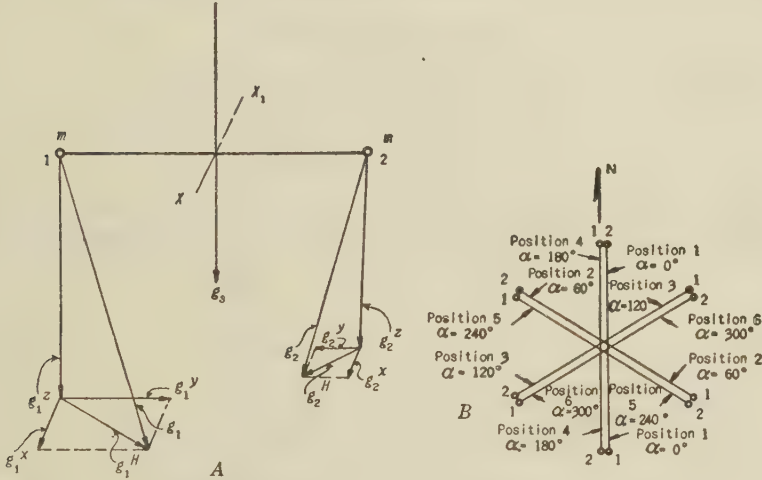


FIGURE 8.—A, Forces on Coulomb balance; B, plan of positions of single balance for solution of balance equation

and 300° . (See fig. 8, B.) The various known quantities are listed in the following table.

Position	1	2	3	4	5	6
Azimuth, α	0°	60°	120°	180°	240°	300°
Reading, θ	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
$\cos \alpha$	1	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	$\frac{1}{2}$
$\sin \alpha$	0	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
$\cos 2\alpha$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$
$\sin 2\alpha$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$

$$A = \frac{K}{r}, \quad B = \frac{mbh}{r}.$$

Substituting in (46) and solving the resulting equation:

$$\theta_1 - \theta_0 = AV_{xy} + BV_{zy},$$

$$\theta_2 - \theta_0 = \frac{A\sqrt{3}}{4}(V_{yy} - V_{xx}) - \frac{A}{2}V_{xy} - \frac{B\sqrt{3}}{2}V_{xx} + \frac{B}{2}V_{zy},$$

$$\theta_3 - \theta_0 = -\frac{A\sqrt{3}}{4}(V_{yy} - V_{xx}) - \frac{A}{2}V_{xy} - \frac{B\sqrt{3}}{2}V_{xx} - \frac{B}{2}V_{zy},$$

$$\theta_4 - \theta_0 = AV_{xy} - BV_{zy},$$

$$\theta_5 - \theta_0 = \frac{A\sqrt{3}}{4} (V_{yy} - V_{xx}) - \frac{A}{2} V_{xy} + \frac{B\sqrt{3}}{2} V_{xz} - \frac{B}{2} V_{zy}$$

$$\theta_6 - \theta_0 = -\frac{A\sqrt{3}}{4} (V_{yy} - V_{xx}) - \frac{A}{2} V_{xy} + \frac{B\sqrt{3}}{2} V_{xz} + \frac{B}{2} V_{zy}$$

Solution:

$$\theta_0 = \frac{\theta_1 + \theta_3 + \theta_5}{3} = \frac{\theta_2 + \theta_4 + \theta_6}{3},$$

$$(V_{yy} - V_{xx}) = \frac{1}{A\sqrt{3}} (\theta_2 - \theta_3 + \theta_5 - \theta_6),$$

$$V_{xy} = \frac{1}{6A} (3\theta_1 - 2\theta_2 + \theta_4 - 2\theta_6),$$

$$V_{xz} = \frac{1}{2B\sqrt{3}} (\theta_5 + \theta_6 - \theta_2 - \theta_3),$$

$$V_{zy} = \frac{1}{2B} (\theta_2 - \theta_3 - \theta_5 + \theta_6) = \frac{1}{2B} (\theta_1 - \theta_4).$$

Many instruments are now being constructed with two complete balance systems spaced 180° apart (fig. 9, A). Thus, for each posi-

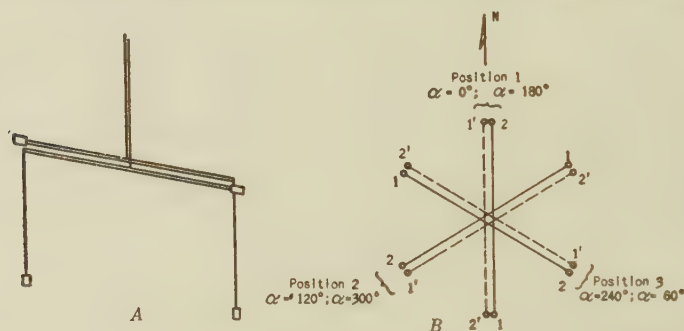


FIGURE 9.—A, Duplex-balance system; B, plan of positions of duplex balance for solution of balance equation

tion of the balance two sets of readings are taken. In this case there are two unknown zero readings, making six unknowns in all; but since two equations are obtained for each position of the balance, only three settings are necessary, corresponding to azimuths of 0° , 120° , and 240° for one system and 180° , 300° , and 60° for the other (see fig. 9, B).

The known quantities are given below:

Balance system	No. 1			No. 2		
Position.....	1	2	3	1	2	3
Azimuth, α	0	120°	240°	180°	300°	60°
Reading, θ	θ_1	θ_2	θ_3	θ_1'	θ_2'	θ_3'
$\cos \alpha$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	-1	$\frac{1}{2}$	$\frac{1}{2}$
$\sin \alpha$	0	$\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$\cos 2\alpha$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$
$\sin 2\alpha$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$

$\theta_0 = 0$ position of balance 1.
 $\theta_0' = 0$ position of balance 2.

$$A = \frac{K}{\tau}, B = \frac{mbh}{\tau}, \text{ balance system 1.}$$

$$A' = \frac{K'}{\tau'}, B' = \frac{m'b'h'}{\tau'}, \text{ balance system 2.}$$

$$D = A'B + AB'.$$

When substituted in (46) the six equations are as follows:

$$\theta_1 - \theta_0 = AV_{xy} + BV_{zy},$$

$$\theta_2 - \theta_0 = -A\frac{\sqrt{3}}{4}(V_{yy} - V_{xx}) - \frac{A}{2}V_{xy} - B\frac{\sqrt{3}}{2}V_{xz} - \frac{B}{2}V_{zy},$$

$$\theta_3 - \theta_0 = A\frac{\sqrt{3}}{4}(V_{yy} - V_{xx}) - \frac{A}{2}V_{xy} + B\frac{\sqrt{3}}{2}V_{xz} - \frac{B}{2}V_{zy},$$

$$\theta_1' - \theta_0' = A'V_{xy} - B'V_{zy},$$

$$\theta_2' - \theta_0' = -A'\frac{\sqrt{3}}{4}(V_{yy} - V_{xx}) - \frac{A'}{2}V_{xy} + B'\frac{\sqrt{3}}{2}V_{xz} + \frac{B'}{2}V_{zy},$$

$$\theta_3' - \theta_0' = A'\frac{\sqrt{3}}{4}(V_{yy} - V_{xx}) - \frac{A'}{2}V_{xy} - B'\frac{\sqrt{3}}{2}V_{xz} + \frac{B'}{2}V_{zy}.$$

Solution:

$$\theta_0 = \frac{\theta_1 + \theta_2 + \theta_3}{3}, \quad \theta_0' = \frac{\theta_1' + \theta_2' + \theta_3'}{3},$$

$$(V_{yy} - V_{xx}) = \frac{2}{\sqrt{3}D}[B'(\theta_3 - \theta_2) + B(\theta_3' - \theta_2')],$$

$$V_{xy} = -\frac{1}{3D}[B'(\theta_2 + \theta_3 - 2\theta_1) + B(\theta_2' + \theta_3' - 2\theta_1')],$$

$$V_{xz} = \frac{1}{\sqrt{3}D}[A'(\theta_3 - \theta_2) - A(\theta_3' - \theta_2')],$$

$$V_{zy} = -\frac{1}{3D}[A'(\theta_2 + \theta_3 - 2\theta_1) - A(\theta_2' + \theta_3' - 2\theta_1')].$$

These solutions make possible computation of the various unknowns from the balance readings and constants. As the values of the second partial derivatives are known, the gradient and curvature function may be found from equations (13) and (36), respectively.

DIMENSIONS AND UNITS

The potential of a gravity field is defined in terms of work. Its dimensions are therefore force times distance, or in terms of fundamental C. G. S. units:

$$V = [\text{grams}] [\text{centimeters}]^2 [\text{seconds}]^{-2}.$$

The first differential of V gives force, and therefore:

$$g = [\text{grams}] [\text{centimeters}] [\text{seconds}]^{-2}.$$

Differentiating force, the gradient is obtained with dimensions as follows:

$$\frac{\partial g}{\partial s} = Gr_g = [\text{grams}] [\text{seconds}]^{-2}.$$

The curvature function was defined as $g \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right)$. Hence,

$$g \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = [\text{grams}] [\text{centimeters}] [\text{seconds}]^{-2} \times [\text{centimeters}]^{-1},$$

or,

$$g \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) = [\text{grams}] [\text{seconds}]^{-2}.$$

Thus, both the gradient and curvature function have the same dimensions and are measured in terms of the same unit. This unit is known as the Eötvös unit (E) and is defined as 1×10^{-9} dynes per centimeter.

Suppose, for example, that a total uniform horizontal gradient of 30 E is measured at a given station with a balance having a beam 30 cm long. This will give a difference in the force of gravity between the two ends of the balance system of 9×10^{-7} dynes, or about 1 part in 100 million.

Similarly, for a curvature function of 30 E, $\left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right)$ has a value of $3 \times 10^{-11} (\text{cm})^{-1}$. Thus, if ρ_2 is taken as the mean radius of the earth, approximately 6.37×10^8 cm, $(\rho_2 - \rho_1)$ will equal 0.12×10^8 cm, or approximately 75 miles. However, if the field configuration is such that $\rho_2 = 2,000$ miles, a curvature function of 30 E corresponds to a $\rho_2 - \rho_1$ of only 19.3 miles, whereas if $\rho_2 = 1,000$ miles, $(\rho_2 - \rho_1)$ would be 4.8 miles. Thus, for a curvature function of given magnitude the difference between maximum and minimum radii of curvature varies as the product of the two radii of curvature. Physically, then, the greater the curvature of the equipotential surfaces of gravity the smaller the relative difference between maximum and minimum radii of curvature the torsion balance can detect. Thus, for $\rho_2 = 4,000$ miles a curvature function of 30 E gives $(\rho_2 - \rho_1) = 77$ miles; if $\rho_2 = 400$ miles the same curvature function (30 E) gives $(\rho_2 - \rho_1) = 0.77$ mile—that is, one-hundredth the difference or 10 times the sensitivity in the first case.

It is of interest to determine the vertical gradient of gravity due to elevation alone. This is found to be 3,080 E. The maximum normal horizontal gradient due to rotation and spheroidal shape of the earth occurs at a latitude of 45° and is equal to 8.15 E, or only about 0.0027 times the vertical gradient due to elevation. As the latitude increases or decreases from 45° this normal horizontal gradient diminishes, reaching zero at the poles and equator.

DETERMINATION OF INSTRUMENT CONSTANTS

The sensitivity of torsion balances at present ranges from 2 to 4 Eötvös units, and in general gradients or curvature functions seldom exceed 150 E. Consequently, the possible observational error is well over 1 per cent. It follows that an accuracy of 1 per cent in the determination of the torsion-balance constants is entirely satisfactory.

DETERMINATION OF m AND m'

The masses of weights m and m' are found by weighing each separately on a suitable balance scale. To avoid possible changes in

the size of m and m' due to chemical action, particularly oxidation, the weights are usually gold or platinum.

DETERMINATION OF b AND h

The half-length of the balance beam, the length of the suspension of the lower weight, and the length of the light beam used to record or indicate the position of the suspended system are measured with an ordinary rule or scale.

DETERMINATION OF K AND τ

The moment of inertia of the balance system K can be computed if the various physical dimensions are known. However, an experimental determination is easily made. Body S , having a known moment of inertia, K_s , is placed on a suitable torsion wire, and period of oscillation T_s is observed. Standard object S is then removed and balance system K substituted. The new period of oscillation T is observed, and the moment of inertia of balance system K is found from the relation:

$$K = K_s \left(\frac{T}{T_s} \right)^2.$$

Then, the torsion constant of balance wire τ may be found by observing the period of oscillation of the balance, whence

$$\tau = 4\pi^2 \frac{K}{T^2}.$$

Instead of the balance beam and weights a standard body of known moment of inertia, K_s , may be used on the torsion wire, the same formula applying when K_s replaces K .

An alternate method of finding τ was used by Eötvös and is illustrated in Figure 10, A. A large spherical mass, M , is brought close to one of the weights of the balance. The lower weight is usually used, as it reduces the effect of M on the balance beam.

If

M = mass of calibrating weight, grams,
 m = mass of balance weight, grams,
 D = distance between centers of M and m , centimeters,
 G = gravitational constant = 6.66×10^{-8} ,
 b = one-half length of balance beam, centimeters,
 θ = angular displacement of balance system due to M , radians,

then

$$\tau\theta = G \frac{mMb}{D^2},$$

whence

$$\tau = 6.66 \frac{mMb}{\theta D^2} \times 10^{-8}.$$

As an example of the mass M required, assume that a deflection of $2'$ is desired, distance D being 2 cm. Then, for a balance having $m = 3.6$ grams, $b = 11$ cm, and $\tau = 0.06$, a mass of 54.7 grams would be required. Again, if $m = 25$ grams, $b = 20$ cm, and $\tau = 0.5$, $M = 36.3$ grams.

Quantity τ is a function of the dimensions of the torsion wire and the torsion modulus of the material.

$$\tau = \frac{\pi a^4}{2 L} \mu.$$

If μ , a , and L are known τ may be calculated. However, an experimental determination of τ is preferable, as it takes into account any slight irregularities in the wire.

In making the experimental determinations of K and τ by observing periods of oscillation, as described, Jung³ points out that an appreciable change in the horizontal force of gravity may cause a slight error in the results. This force would have the effect of a change in τ . Hence

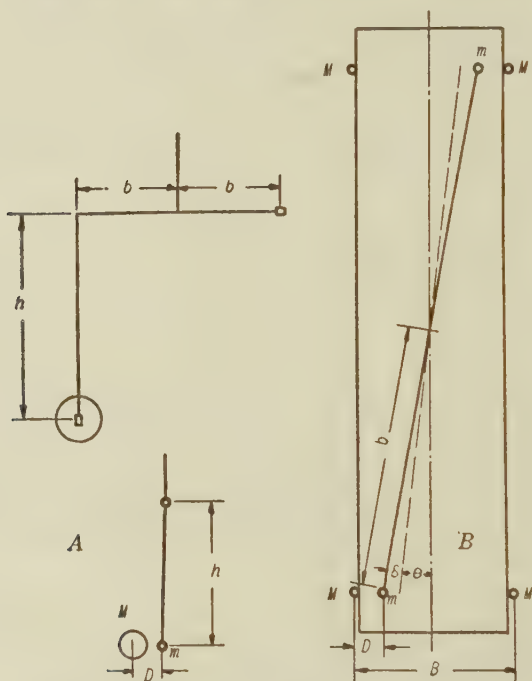


FIGURE 10.—A, Method of determining K and τ ; B, effect of mass of balance housing

by using a standard torsion wire having a large value of τ this effect is made negligible.

SENSITIVITY OF BALANCE

In speaking of the sensitivity of the torsion balance one should distinguish between gradient sensitivity and curvature sensitivity. The former is expressed as the maximum angular deflection produced by a gradient of 1 E, whereas the latter is designated as the maximum angular deflection caused by a curvature function of 1 E.

For torsion balances having similar suspended systems the sensitivity varies inversely as τ , the torsion constant of the wire, and directly as the square of T , the period of oscillation of the torsion system.

³ Jung, Karl, *Gravimetrische Methoden der angewandten Geophysik*: Wien-Harms Handbuch der Experimental-Physik, vol. 25 (Geophysik), pt. 3 (Angewandte Geophysik), Leipzig, 1930, pp. 49-208.

These facts are demonstrable from solutions of the equations of a torsion pendulum.

The differential equation for a torsion pendulum in a resistanceless medium is given as

$$K \frac{d^2\theta}{dt^2} = -\frac{\pi}{2} \mu \frac{a^4}{L} \theta = -\tau \theta. \quad (47)$$

The solution of equation (47) gives for period T :

$$T = 2\pi \sqrt{\frac{K}{\tau}}. \quad (48)$$

Let

F_G = torque produced on balance system by a gradient of 1 E.

F_C = torque produced on balance system by a curvature function of 1 E.

Then the respective sensitivities θ_G and θ_C are given as

$$\theta_G = \frac{F_G}{\tau}, \quad \theta_C = \frac{F_C}{\tau},$$

or, in general,

$$\theta_S = \frac{F_S}{\tau}, \quad (49)$$

where θ_S may be either θ_G or θ_C .

Solving equation (49) for τ and substituting in (48):

$$T = 2\pi \sqrt{\frac{K\theta_S}{F_S}},$$

or

$$\theta_S = \frac{T^2 F_S}{4\pi^2 K}. \quad (50)$$

Torques F_G and F_C can be expressed in terms of the torsion-balance quantities. Referring to equation (46), the following relations express the torques for gradient and curvature function of 1 E, respectively:

$$F_G = mbh \times 10^{-9},$$

$$F_C = \frac{K}{2} \times 10^{-9}.$$

Or, expressed as sensitivities:

$$\left. \begin{aligned} \theta_G &= \frac{mbh}{\tau} \times 10^{-9} \\ \theta_C &= \frac{K}{2\tau} \times 10^{-9} \end{aligned} \right\} \begin{aligned} &\text{if } K \approx 2b^2m \\ &\frac{\theta_G}{\theta_C} = \frac{h}{b}. \end{aligned}$$

For example, a balance in which $\frac{K}{\tau} = 60,000$ and $\frac{mbh}{\tau} = 89,000$ would have sensitivities of

$$\theta_G = 89,000 \times 10^{-9} = 18.3 \text{ seconds per 1 E horizontal gradient.}$$

$$\theta_C = 30,000 \times 10^{-9} = 6.2 \text{ seconds per 1 E curvature function.}$$

Again, for a balance in which $\frac{K}{\tau} = 40,000$ and $\frac{mbh}{\tau} = 55,000$

$$\theta_G = 55,000 \times 10^{-9} = 11.3 \text{ seconds per 1 E horizontal gradient.}$$

$$\theta_C = 20,000 \times 10^{-9} = 4.1 \text{ seconds per 1 E curvature function.}$$

Evidently, any desired ratio of $\frac{\theta_G}{\theta_C}$ may be obtained by properly

adjusting the values of h and b . For example, if $\frac{\theta_G}{\theta_C}$ is to equal unity, h must equal b .

The period of free oscillation for the balance system in terms of gradient and curvature sensitivities is

$$T = 2\pi \sqrt{\frac{2b\theta_G}{h} \times 10^9} = 2\pi \sqrt{2\theta_C \times 10^9}.$$

Thus, for a balance having a gradient sensitivity of 18.3 seconds, with a corresponding curvature sensitivity of 6.2 seconds, the period will be 1,540 seconds.

Conversely, if a balance is to be constructed with a ratio of $\frac{b}{h} = 0.5$ and a period of 20 seconds, the gradient sensitivity will be approximately 10^{-8} radian, or 0.002 second of arc, corresponding to a curvature sensitivity of 0.001 second of arc.

EFFECT OF MASS OF INSTRUMENT

The effect of the mass of the instrument on the deflections of the balance system will now be investigated.

For convenience in computing this effect the mass of the balance is divided into three parts: (1) The mass that rotates through the same azimuth as the balance system ($\theta + \alpha$); (2) the mass that rotates through the angle α ; and (3) the fixed mass.

The first type includes, primarily, the balance beam and the lower weight suspension. As these masses always maintain the same relative position with regard to the suspended system, their effect is a constant at all times and does not produce any change in the results, inasmuch as all quantities are computed from differences between readings and thereby eliminate such constant deflections.

The second type of mass includes the balance housing, recording apparatus, and any other mass that is rotated through angle α .

The suspended system rotates through angle θ relative to these masses, and their effect is to increase θ by an additional small angle, δ . The problem is to determine a relation between δ and θ for a balance of a given size.

Since the force of attraction between two objects varies as the inverse square of their distance of separation, for all practical purposes only those portions of the case within a few centimeters of the two balance weights need be considered. Assume these parts of the case to be as four equal concentrated, effective masses, M , two on each side of the balance weights, as indicated in Figure 10, *B*. Then if

- b = one-half length of balance arm,
- B = breadth of balance housing,
- D = distance from m to nearest M ,
- θ = corrected angular deflection,
- δ = angular deflection caused by M ,
- G = gravitational constant = 6.66×10^{-8} ,
- τ = rigidity of wire,

$$\tau\delta = 2GmMb \left[\frac{1}{D^2} - \frac{1}{(B-D)^2} \right] \cos(\theta + \delta),$$

but $\cos(\theta + \delta) \approx 1$, since $(\theta + \delta)$ is always less than 1° .
Hence

$$\delta = \frac{2GmMb}{\tau} \left[\frac{1}{D^2} - \frac{1}{(B-D)^2} \right].$$

To determine the rate of change of δ with θ differentiate δ with respect to θ through D , using the approximate relation $D = \frac{B}{2} - b\theta$.

Therefore

$$\begin{aligned}\frac{dD}{d\theta} &= -b \\ \frac{d\delta}{d\theta} &= \frac{d\delta}{dD} \frac{dD}{d\theta} = -b \frac{d\delta}{dD} \\ \frac{d\delta}{d\theta} &= \frac{4mb^2MG}{\tau} \left[\frac{1}{D^3} + \frac{1}{(B-D)^3} \right] \\ 2mb^2 &= K.\end{aligned}$$

Hence, the absolute value of $\frac{d\delta}{d\theta}$, denoted by $\left| \frac{d\delta}{d\theta} \right|$, is found to be

$$\left| \frac{d\delta}{d\theta} \right| = 2G \frac{K}{\tau} M \left[\frac{1}{D^3} + \frac{1}{(B-D)^3} \right].$$

This quantity should be kept as small as possible. As a numerical example, consider a balance in which $\frac{K}{\tau} = 60,000$ C. G. S. units and $M = 12$ grams. Then

$$\left| \frac{d\delta}{d\theta} \right| = 0.096 \left[\frac{1}{D^3} + \frac{1}{(B-D)^3} \right].$$

If $B = 8$ cm and $D = 3$ cm, $\left| \frac{d\delta}{d\theta} \right| = 0.0043$; if $D = 2$ cm, $\left| \frac{d\delta}{d\theta} \right| = 0.012$.

Again, if $B = 6$ cm and $D = 2$ cm, $\left| \frac{d\delta}{d\theta} \right| = 0.012$; if $D = 1$ cm, $\left| \frac{d\delta}{d\theta} \right| = 0.096$.

It follows that quantity D is the controlling factor in the ratio. By keeping D 3 to 4 cm $\left| \frac{d\delta}{d\theta} \right|$ is kept small. Physically, B would be 8 cm or more, and at the same time the torsion wire would be kept in such a position that the balance weights would be approximately centered with respect to the sides of the balance casing. For example, the maximum deflection likely to be encountered in practice is of the order of 2,000 seconds of arc. If $B = 8$ cm and $D = 3.5$ cm, the error angle δ would be

$$\delta = \left| \frac{d\delta}{d\theta} \right| \theta = 0.0034 \times 2,000 = 6.8 \text{ seconds of arc.}$$

This angle would be entirely negligible as far as all practical results are concerned.

In a discussion of the effect of the instrument base and other similar parts which do not rotate, it is necessary to develop the relations for the effect of a mass point on the various quantities of the gravity field. (See fig. 11, A.)

Let M = mass concentrated at any point in space (ξ, η, ζ) .

P = point (x, y, z) at which the effect of M is to be computed.

$r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}$ = distance from M to P .

G = gravitational constant = 6.66×10^{-8} , C. G. S. units.

F = force on unit mass at P due to M .

$$F = \frac{GM}{r^2}.$$

The three components of F are found to be:

$$F_x = \frac{GM}{r^2} \cos \psi \sin \omega = \frac{GM \sqrt{(x-\xi)^2 + (y-\eta)^2}}{r^2} \times \frac{x}{\sqrt{(x-\xi)^2 + (y-\eta)^2}}$$

$$F_x = \frac{GMx}{r^3}.$$

Similarly,

$$F_y = \frac{GM}{r^2} \cos \psi \cos \omega = \frac{GM y}{r^3},$$

$$F_z = \frac{GM}{r^2} \sin \psi = \frac{GM z}{r^3}.$$

The various second partial derivatives of V at P are obtained directly from these components by single integrations. Thus

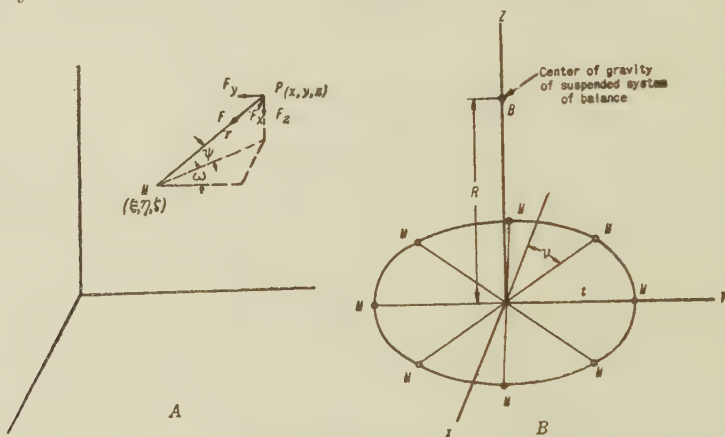


FIGURE 11.—A, Computation of gravity field for a point mass; B, effect of mass of instrument stand

$$V_{yy} = \frac{\partial F_y}{\partial y} = MG \left[\frac{r^2 - 3 \frac{(y-\eta)^2}{r^2}}{r^5} \right],$$

$$V_{xx} = \frac{\partial F_x}{\partial x} = MG \left[\frac{r^2 - 3 \frac{(x-\xi)^2}{r^2}}{r^5} \right].$$

Whence

$$(V_{yy} - V_{xx}) = 3MG \frac{(x-\xi)^2 - (y-\eta)^2}{r^5},$$

$$V_{xy} = 3MG \frac{(x-\xi)(y-\eta)}{r^5},$$

$$V_{zx} = 3MG \frac{(z-\zeta)(x-\xi)}{r^5},$$

$$V_{zy} = 3MG \frac{(z-\zeta)(y-\eta)}{r^5}.$$

These equations express the effects of single mass M in space. The base and other fixed portions of the instrument may be regarded as a series of such masses in space, the total effect being a simple summation of the individual effects. In general, the stationary parts of a torsion balance are symmetrical about the vertical (Z) axis; therefore the mass may be considered distributed in a symmetrical form about this axis, either as a series of individual mass points or as a uniform mass ring. The problem is then to determine the effect of such mass

distribution at the center of gravity of the suspended system, that is, at a point on the Z axis.

Consider a series of n individual points, each of mass M , concentrated at each point. Let these points be spaced at equal intervals around the circumference of a circle of radius t whose plane is perpendicular to the Z axis and whose center lies on that axis at distance R below the center of gravity of the suspended system of the balance.

(See fig. 11, B .) Then the n particles make angles of $\nu, \nu + \frac{2\pi}{n}, \nu + \frac{4\pi}{n} \dots$

$\nu + \frac{2(n-1)\pi}{n}$ with a reference axis. For point A the coordinates are:

$$x = t \cos \left(\nu + \frac{2\pi}{n} \right),$$

$$y = t \sin \left(\nu + \frac{2\pi}{n} \right),$$

$$z = -R,$$

and similarly for the other points.

The coordinates of B are $0, 0, 0$; that is, the origin is at B .

Summing the effects of the various points at B :

$$(V_{yy} - V_{xx}) = \frac{3MGt^2}{(t^2 + R^2)^{\frac{5}{2}}} \left\{ \cos^2 \nu + \cos^2 \left(\nu + \frac{2\pi}{n} \right) + \dots + \cos^2 \left[\nu + \frac{2(n-1)\pi}{n} \right] - \sin^2 \nu - \sin^2 \left(\nu + \frac{2\pi}{n} \right) - \dots - \sin^2 \left[\nu + \frac{2(n-1)\pi}{n} \right] \right\},$$

or

$$(V_{yy} - V_{xx}) = \frac{3MGt^2}{(t^2 + R^2)^{\frac{5}{2}}} \left\{ \sum_{r=0}^{r=(n-1)} \left[\cos^2 \left(\nu + \frac{2\pi r}{n} \right) - \sin^2 \left(\nu + \frac{2\pi r}{n} \right) \right] \right\}.$$

Finally,

$$(V_{yy} - V_{zz}) = \frac{3MGt^2}{(t^2 + R^2)^{\frac{5}{2}}} \left\{ \cos 2\nu \sum_{r=0}^{r=(n-1)} \cos \frac{4\pi r}{n} - \sin 2\nu \sum_{r=0}^{r=(n-1)} \sin \frac{4\pi r}{n} \right\}.$$

Similarly,

$$V_{xy} = \frac{3}{2} \frac{MGt^2}{(t^2 + R^2)^{\frac{5}{2}}} \left\{ \sin 2\nu \sum_{r=0}^{r=(n-1)} \cos \frac{4\pi r}{n} + \cos 2\nu \sum_{r=0}^{r=(n-1)} \sin \frac{4\pi r}{n} \right\},$$

$$V_{xz} = -\frac{3MGtR}{(t^2 + R^2)^{\frac{5}{2}}} \left\{ \cos \nu \sum_{r=0}^{r=(n-1)} \cos \frac{2\pi r}{n} - \sin \nu \sum_{r=0}^{r=(n-1)} \sin \frac{2\pi r}{n} \right\},$$

$$V_{yz} = -\frac{3MGtR}{(t^2 + R^2)^{\frac{5}{2}}} \left\{ \sin \nu \sum_{r=0}^{r=(n-1)} \cos \frac{2\pi r}{n} + \cos \nu \sum_{r=0}^{r=(n-1)} \sin \frac{2\pi r}{n} \right\}.$$

A study of these formulas reveals the following facts:

If

$$n \geq 3, \sum_{r=0}^{r=(n-1)} \cos \frac{4\pi r}{n} = 0, \text{ and } \sum_{r=0}^{r=(n-1)} \sin \frac{4\pi r}{n} = 0.$$

If

$$n \geq 2, \sum_{r=0}^{r=(n-1)} \cos \frac{2\pi r}{n} = 0, \text{ and } \sum_{r=0}^{r=(n-1)} \sin \frac{2\pi r}{n} = 0.$$

Hence, if $n \geq 3$ the curvature function is zero, whereas if $n \geq 2$ the gradient is zero; that is, physically, if the fixed parts of the balance are shaped so that they are the equivalent of three or more symmetrically placed equal masses they will not affect the gradient or curvature at the center of gravity of the suspended system and hence will not affect values of gradient and curvature computed from torsion-balance operations.

A summary of the remarks on the effect of the instrument mass shows that by proper construction all such effects can be reduced to negligible magnitudes. Actually, the only object that produces a resultant correction is the mass of the balance housing, and the author has shown that by proper design this can be made negligible.

TORSION FIBERS

In the final analysis satisfactory performance of the Eötvös torsion balance depends primarily on the use of a torsion fiber with stable physical properties. The paramount importance of this factor is amply attested to by the time and money expended on investigations of fibers of various types, their preparation and behavior under different operating conditions.

Drawing out a fiber sets up internal stresses which cause apparent inhomogeneities in the material. Such a suspension placed under load exhibits an unstable zero position and an erratic behavior with temperature changes. In addition, a hysteresis effect is present. Thus, if a fiber is subjected to a series of alternating stresses, after these forces are removed, it will show a series of very minute alternating displacements similar to the impressed strains.

A study of the proper processing of torsion-balance fibers to assure stable physical properties under load has produced a method of heat treatment that yields satisfactory results. The quantitative features of this method are at present carefully guarded manufacturing secrets, but the essential features are about as follows:

After the fiber has been drawn to the proper diameter it is annealed at a rather high temperature. It is then slowly cooled in the furnace and is "aged" by being placed under load and subjected to a rapid series of heating and cooling periods. This last step is to remove internal stresses and is somewhat analogous to demagnetization of an iron bar by rapidly reversing a constantly decreasing current in an inclosing solenoid.

Torsion fibers for field balances are now made of tungsten or a platinum-iridium alloy.

A comprehensive discussion of torsion fibers would cover many pages, but the author believes that the foregoing brief remarks present some of the difficulties involved and stress the importance of the suspension in the operation of the torsion balance.

INSTRUMENT DIMENSIONS IN PRACTICE

Since the first Eötvös torsion balance was constructed by its originator, Roland von Eötvös, in 1890 numerous manufacturers have produced similar instruments designed largely for field use in geophysical prospecting. Comparison reveals a considerable range of physical dimensions, as illustrated in Table 1, taken from a German text on the subject. The symbols, defined on page 15, are shown in Figure 12.

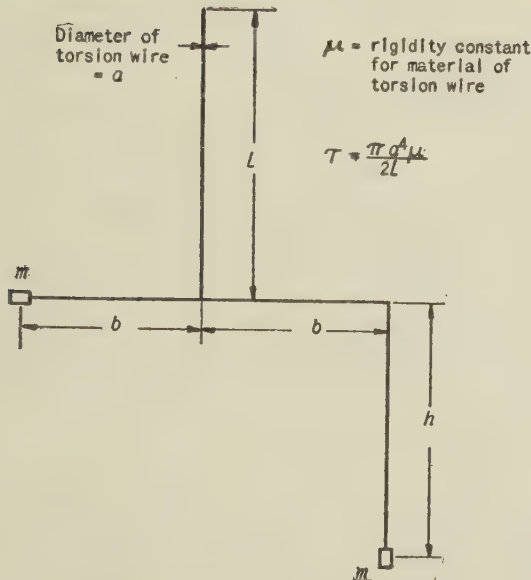


FIGURE 12.—Balance dimensions used in Table 1

TABLE 1.—Balance dimensions¹

m = mass of each balance weight [grams]
 h = length of suspension of lower weight below balance beam [centimeters]
 b = one-half length of balance beam [centimeters]
 a = diameter of torsion wire [microns] [0.001 centimeter]
 L = length of torsion wire [centimeters]
 τ = rigidity constant of torsion wire [centimeters]² [grams] [seconds]⁻²
 K = moment of inertia of balance system about axis of torsion wire [grams] [centimeters]²
 T = period of balance
 θ_s = angular sensitivity of balance [seconds of arc]

Balance	m	h	b	Torsion wire				K^2	θ_s^2		T^2	
				Material	a	L	τ^2		1 E gradient	1 E curvature	Sec- onds	Min- utes
1. Eötvös single balance, 1880.....	30	103	13.2	Pt-Ir.....	40	110	0.23	10,800	36	4.9	1,355	22.6
2. Eötvös single balance, 1898.....	25.4	64.8	20	do.....	40	56	.46	21,100	15	4.8	1,345	22.4
3. Eötvös double balance (small), 1925.....	8.5	31.7	10	do.....	20	50	.035	1,760	16	5	1,405	23.4
4. Eötvös balance (miniature), 1908.....	1.4	20.3	5	{Quartz Pt-Ir.....}	45	.0046		90	6.4	2	875	14.6
5. Schweydar-Askania large balance.....	30.8	61.2	20	Pt-Ir.....	40	56	.49	26,700	16	5.5	1,460	24.4
6. Schweydar-Askania Z-bar balance.....	22.5	45.0	20	do.....	36	28	.47	19,500	8.8	4.3	1,275	21.3
				W.....	26	28	.23	19,500	18	9	1,825	30.4
				W.....	30	28	.37	19,500	11	5.5	1,440	24.0
7. Hecker, small model, 1925.....	17.0	37.3	11.1	Pt-Ir.....	31	50	.15	4,500	9.7	3.1	1,085	18.1
				W.....	21	50	.066	4,500	22	7	1,640	27.4
				W.....	26	50	.12	4,500	12	3.9	1,210	20.2
8. Hecker, large model, 1927.....	18.0	45.2	16.2	Pt-Ir.....	30	50	.12	9,600	23	8	1,770	29.5
				W.....	26	50	.12	9,600	23	8	1,770	29.5
9. Haff, 1928.....	3.6	27	11	Pt alloy.....		.02		1,000	11	5	1,400	23.3
10. Haff, 1929.....	11	33	11	do.....		.06		1,000	3.7	1.8	810	13.5
				do.....	14	10	.045	2,700	18	6	1,540	25.6
11. Oertling.....	25	55	20	Pt-Ir.....	40	60	.050	3,000	18	5.5	1,540	25.6
12. Haack.....	30	50	19	do.....	40	35	.5	22,000	9	5.5	1,315	21.9
13. Tsuboi (quartz glass).....	.25	20	10	Quartz.....	5	20				.6		

¹ Taken from Jung, Karl, Gravimetrische Methoden der angewandten Geophysik: Wien-Harms Handbuch der Experimental-Physik, vol. 25 (Geophysik), pt. 3 (Angewandte Geophysik), Leipzig, 1930, pp. 49-208.

² Approximate values.

Although the sensitivities and periods vary considerably with the make of balance, one feature in regard to the latter is rather striking—in all instances the period is extremely long, ranging from 13.5 to 30.4 minutes in the examples cited.

The average sensitivity of these balances ranges from 2 to about 4 E.

MANIPULATION IN FIELD

A very brief outline of the usual plan of manipulation of the torsion balance in the field will now be presented.

A level area on which to set up the balance is selected whenever circumstances permit. Rocky sites should always be avoided if possible, as hidden boulders cause errors in the readings which are impossible to explain unless their exact locations and sizes are known. The point where the balance is to be placed is smoothed off within a radius of 10 feet, and the instrument bedplate is planted firmly on the ground to afford a stable foundation. If a tent is to be used, it is placed over the bedplate.

The instrument is then placed on its foundation, carefully leveled, and rotated until the balance beam lies along the geographic N-S line. The suspended system is released from its clamping devices, and the balance is left for about two hours to allow the torsion wire to become adjusted to its load so that consistent readings may be obtained. Observations are then begun.

Beginning at the initial position, corresponding to a value for α of 0° , the position of the beam is read from the scale as soon as the suspended system comes to rest. The entire balance is then rotated through the proper angle, and the next reading is taken in a similar manner. This process is repeated until the complete set of data is obtained for the station. It is usually wise to repeat one or two readings to make sure that the observations are consistent.

Each time the balance is rotated the suspended system is set in motion and must come to rest before the steady deflection can be observed. Because of the fine torsion wire which must be used to obtain a visible deflection the restoring force is very small, and the natural period of oscillation of the balance system is usually 10 to 20 minutes. In consequence, after each movement of the instrument a wait of an hour or more may be necessary before a steady deflection is reached.

Balances of the latest type are equipped with clockwork mechanism which automatically rotates the entire instrument through the proper angle at predetermined intervals. The equipment also includes cameras which photograph the scale just before the balance is rotated to its new position. Thus the data for a station are recorded completely, and once the operator has the apparatus set in position he can leave it until the entire set of observations has been completed.

CORRECTION AND INTERPRETATION OF RESULTS

The ultimate object of all geophysical work is location of geologic anomalies or structures beneath the surface of the ground from a determination of the reaction these structures produce on various quantities that can be measured at the surface. In many instances there are no outward indications of the presence of the body; therefore all other effects than those produced by the buried structure must

be removed from the actual readings if the data are to be correctly interpreted.

In torsion-balance work changes in the earth's gravity field are measured at the surface; and after proper corrections for surface conditions and the form and rotation of the earth are applied the resulting net readings are analyzed, and estimates are made as to the general shape and depth of the structures causing these net changes.

In general, torsion-balance corrections are classified under two headings—planetary effects and surface effects.

Planetary effects include gradients and curvatures caused by the fact that the earth is a rotating spheroid. In consequence, a horizontal gradient running north and south from the equator results from variations of the centrifugal force and the distance to the center of mass of the earth. Moreover, as the level surfaces for a homogeneous earth would be concentric spheroids the curvature function will have a value which must be applied as a correction. With the N-S line as reference axis this function will reduce to $(V_{yy} - V_{xx})$, factor V_{xy} being equal to zero.

Both of these corrections have been computed for the various latitudes and are given as a series of tables, thus avoiding the necessity of actually calculating them for every new position on the earth's surface.

The corrections due to surface conditions account for sloping ground, depressions and elevations, and similar objects which may produce gravity effects included in the actual balance readings.

In calculating these effects the resulting formulas are extremely complex. However, by limiting the range over which they apply simplified approximations can be used. To employ these the region about a station is divided into three zones by two concentric circles. Within each of the resulting rings or zones an approximate formula applies with negligible error.

The corrections in the three respective regions, fundamentally the same but different in the magnitude of their effects, are usually classified as: (1) Terrain, (2) topographic, and (3) cartographic.

The terrain effects apply to the region within a circle about 100 feet in radius with the instrument location as a center. The area is usually divided into a number of small segments or zones by means of a series of concentric circles and radial lines. The effect of each segment on the gravity field at the instrument is then calculated. Obviously, if the ground is level and homogeneous the effects of the various zones will annul each other because of their symmetrical location around the station, and no resultant correction will be necessary. On the other hand, if there are irregularities in the specific gravity of the ground or if the surface is uneven, a correction must be made. In doing so the specific gravity and mean elevation of each zone with respect to the instrument must be known. An average density is usually taken, and the elevations are obtained by surveying the region.

The topographic corrections apply to the same type of effects due to features within a circular ring extending from the terrain region to a circle of about 1,000 feet radius about the instrument as a center. Obviously, as the distance from the instrument increases, the size of the object or irregularity which will cause an appreciable disturb-

ance in the gravity field at the instrument must increase as the square of this distance. In level country corrections may sometimes be neglected beyond a certain point.

The cartographic corrections apply to the region beyond the topographic zone. In this territory only large hills or valleys need be considered, and beyond a certain point even these may be disregarded. After an operator attains a fair degree of skill and experience he can tell almost at a glance just how far to go before the effects of ground features become negligible.

To assist calculation of these various corrections, graphical methods are being introduced. The various effects may be determined more easily by charts.

In all of these regions such special features as deep ravines, rivers, buildings, and walls must be taken into account. Their calculation, even in the simplest cases, becomes very involved but approximations can be made, enabling them to be corrected for.

After all of the necessary corrections have been applied to the observed readings of the torsion balance the net results are due to subterranean structures; the problem then is to determine, as far as possible, the location, outline, and depth of these structures.

According to Barton, a correct geological interpretation of the results of a torsion-balance survey depends on the following assumptions:

1. The form, depth, and relative density of a mass causing gradient and curvature anomalies can be determined from the distribution of the gradient and curvature values in the anomalies observed at the surface.
2. There is a direct connection between geologic structure and the anomalous distribution of mass.

Now, at a single point the gradient and curvature anomalies might be due to a small mass relatively close to the surface or a large mass at a great depth. However, if a proper network of stations has been chosen, a complete study of the results of the survey will give a truer interpretation of the conditions as they exist.

Barton, Lancaster-Jones, and others have calculated the effects on the gradient and curvature function of several relatively simple shapes placed at known positions from the point at which the field is observed. From study and comparison of these results the location and character of a hidden body can sometimes be predicted.

Whenever interpretations of any geophysical results are to be made intelligently familiarity with geology is indispensable.

REDUCTION OF TORSION-BALANCE PERIOD

The free period of oscillation of a torsion-balance system is readily reduced by decreasing its moment of inertia about the axis of the suspension or by stiffening the torsion fiber. In either instance the deflection of the balance beam for a given applied force will be reduced. Equation (50) shows that the deflection varies as the square of the period of the suspended system.

The foregoing considerations lead to the obvious conclusion that reduction of the period of the balance means development of a method of measuring extremely minute angular displacements. This will now be discussed.

METHOD OF MEASURING SMALL ANGLES

A system suitable for the measurement of small angular displacements adaptable to the torsion balance should combine simplicity, ruggedness, ease of operation and adjustment, dependability, and extreme sensitivity. After careful study of the problem the system shown diagrammatically in Figure 13 was finally developed.

A source of light and a condenser lens were placed behind a grid or screen with alternate opaque and transparent slits equal in width. This screen (S_1) was distance D from a concave mirror mounted on a torsion rod. By making D equal to the radius of curvature of the mirror a full-size image of screen S_1 was projected on a plane at an equal distance, D , from the mirror. Coincident with this image was placed another screen, S_2 , with alternate opaque and transparent slits of the same width as in S_1 . However, at the center of S_2 an opaque

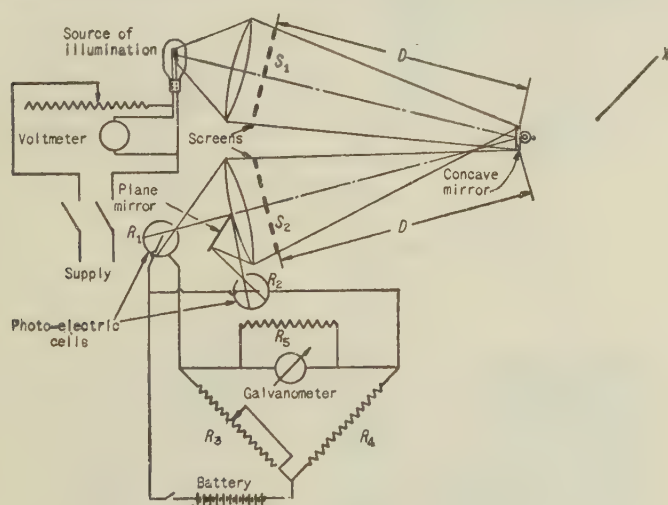


FIGURE 13.—Method of measuring angular displacements

slit of double width was ruled; one-half of the screen was thus put directly out of step with the other half. If the image of S_1 coincided with S_2 over the upper half of the latter so that this portion transmitted the light reaching it through S_1 , the image of S_1 was therefore exactly out of step over the lower half of S_2 ; hence over this portion S_2 excluded all light reaching it through S_1 .

The light transmitted through each half of S_2 was focused on a photo-electric cell, and the two cells were made the ratio arms of a Wheatstone's bridge (see Fig 13). Then by keeping R_4 constant and varying R_3 the bridge was balanced. If the mirror was now given a small angular displacement from its initial position, the image of S_1 was shifted relative to S_2 ; an additional amount of light then passed through one half of S_2 , while a portion of the light through the other half was cut off. This changed the resistances of the two cells—one increasing and the other decreasing—and it was necessary to change R_3 to obtain a balance of the bridge again. Thus the change in resistance R_3 indicated the angular displacement of the torsion rod.

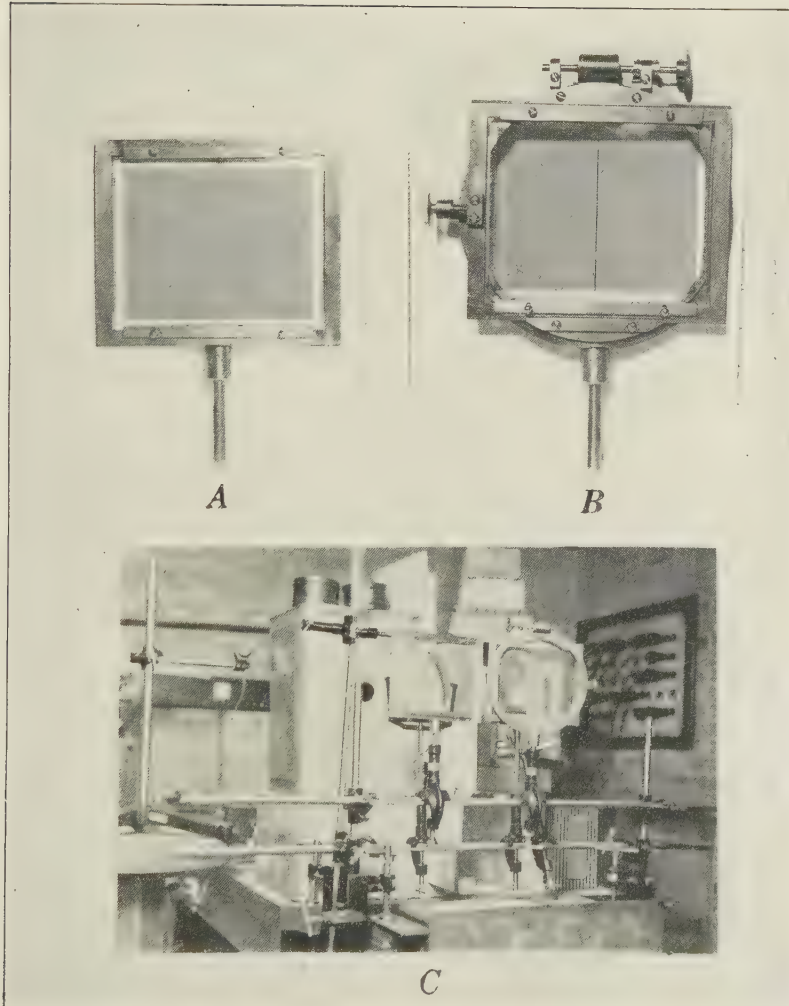


FIGURE 15.—*A*, Plain screen, *S*₁; *B*, split screen, *S*₂; *C*, screens mounted in position

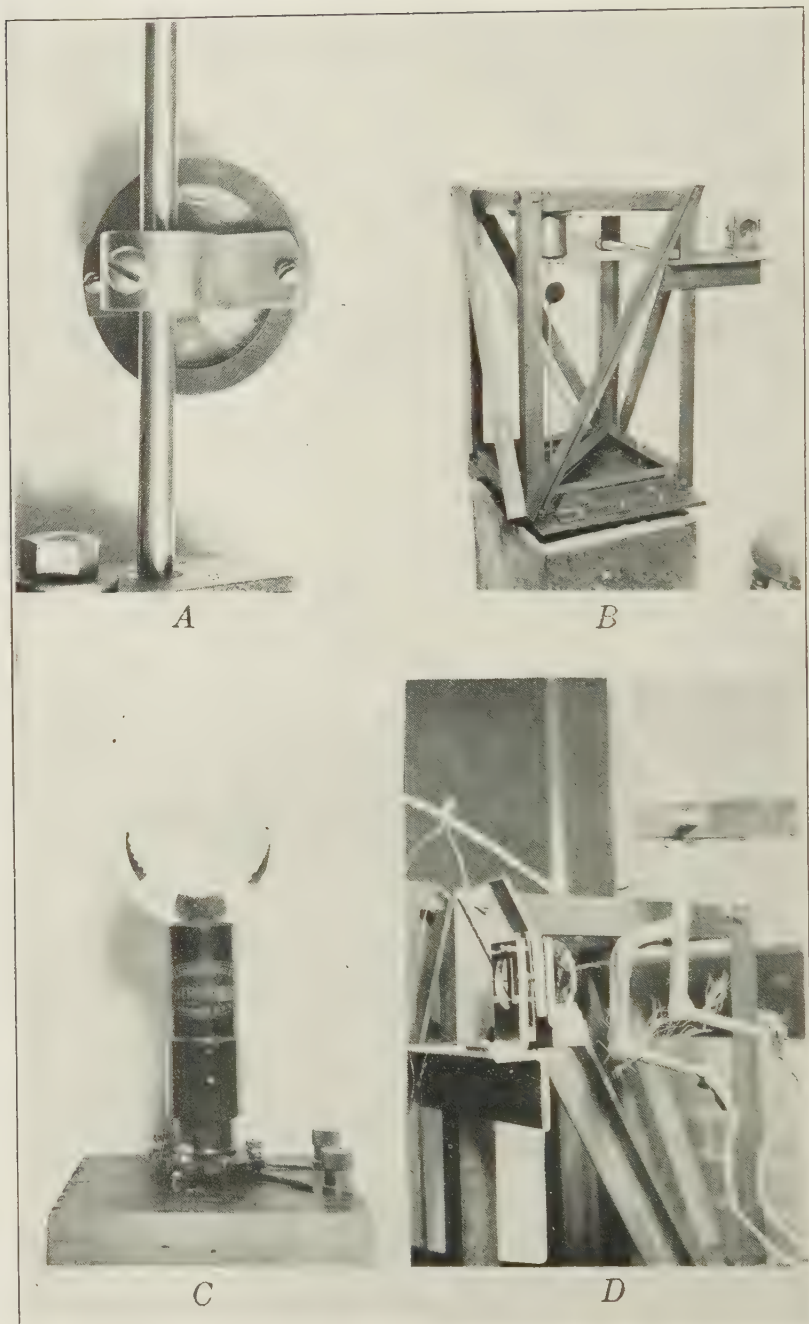
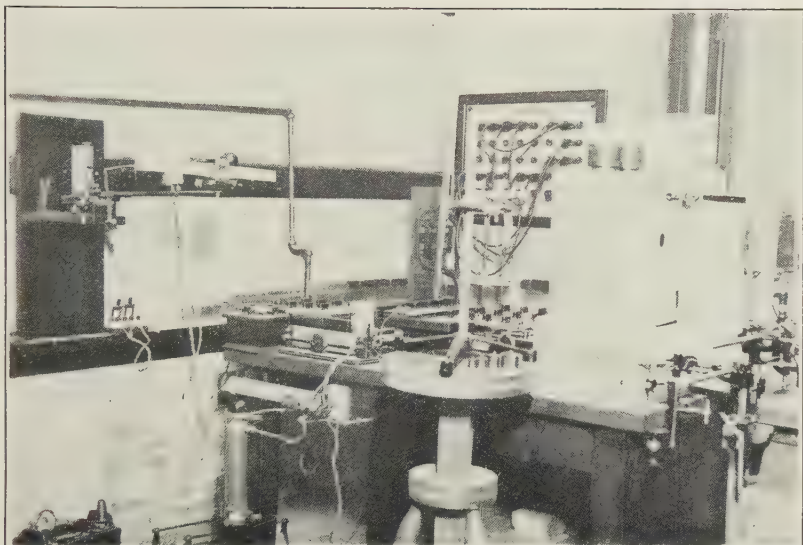
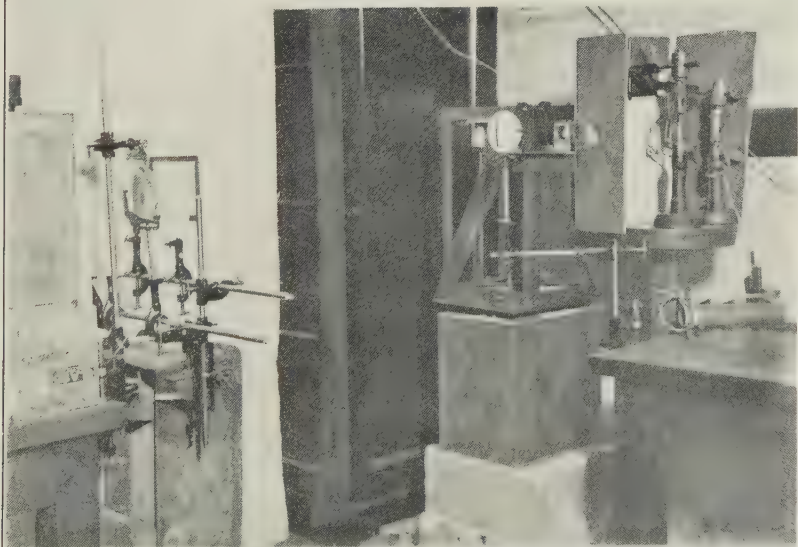


FIGURE 16.—A, Details of mirror mounting; B, torsion frame; C, photo-electric cell; D, interferometer, color filter, and mercury-arc lamp in position



A



B

FIGURE 17.—*A* and *B*, General views of apparatus

A concave mirror $1\frac{1}{2}$ inches in diameter with a 1-meter radius of curvature was mounted in a holding frame which was in turn fastened to a collar riding on the torsion rod. Figure 16, *A*, shows details of this mounting and Figure 16, *B*, the complete assembly of the frame.

The photo-electric tubes were potassium cells of the central-ring anode type with 3-inch-diameter bulbs. Figure 16, *C*, shows one cell mounted in its socket.

The bridge circuit was built up of laboratory 10,000-ohm decade resistance boxes. Resistance R_4 was kept constant at 10,000 ohms, and R_3 was varied to obtain a balance. Voltage was supplied by dry *B* batteries, and a Leeds & Northrup D'Arsonval type *R* wall galvanometer was used as a detector. This instrument had a resistance of 895 ohms, a period of 38 seconds, and a measured sensitivity of 1×10^{-11} ampere per millimeter scale deflection at one-half meter scale distance. It was shunted with 122,000 ohms throughout the work.

The source of illumination for the measuring system was a 500-watt, 120-volt, clear, projection-type lamp in a properly ventilated metal box, with a hole cut in one side to emit the light beam. When operated at full voltage the light intensity was found to be much greater than necessary; therefore the voltage was reduced to 70 for normal operation. However, tests were also made to determine the effect on the bridge characteristics of varying the light intensity.

For purposes of calibration a Fabry-Perot interferometer was used. One plate was mounted on the lever arm of the torsion system 35 cm from the axis of the torsion rod, and the other plate was fastened rigidly to the frame. By making the plates rigorously parallel and using the mercury green line ($\lambda = 5,461$ Ångströms) as a source of illumination a system of Haidinger's fringes was obtained. A telescope cross hair was focused on the field of fringes so that any change in their position could be observed to the nearest tenth part of a fringe. Any variation in the distance of separation of the two plates brought about by a displacement of the lever arm caused a shift of the interference fringes; by observing this change the angular displacement of the torsion rod could be computed by means of the relation:

$$\theta = \frac{\delta}{d} = \frac{n\lambda}{2d},$$

where

θ = angular displacement of torsion rod, radians,
 d = distance from axis of torsion rod to center of interferometer, centimeters,
 δ = change in distance between plates, centimeters,
 λ = wave length of light, centimeters, and
 n = number of fringes passing cross hair.

$$\begin{aligned} d &= 35 \text{ cm, } \lambda = 5.461 \times 10^{-5} \text{ cm,} \\ \theta &= 7.8n \times 10^{-7} \text{ radians.} \end{aligned} \tag{51}$$

Thus, a shift of one fringe of the mercury green line corresponded to an angular displacement of the torsion system of 7.8×10^{-7} radians. In this manner the apparatus was calibrated.

The interferometer, color filter, and mercury arc are shown in Figure 16, *D*, and general views of the complete apparatus, including clamping arrangements and bridge circuits, in Figure 17, *A* and *B*.

EXPERIMENTAL DIFFICULTIES

One of the gravest difficulties to be overcome in realizing satisfactory experimental results was mechanical vibration of the apparatus by rotating machinery, shocks, and other similar disturbing factors in the laboratory. Such vibrations, when transmitted to the torsion frame or optical system, caused the bridge balance to be extremely unsteady and made it impossible to obtain any quantitative data from the apparatus. To avoid these disturbances all observations were taken late at night, and the torsion frame and optical devices were mounted on bases made of bricks, cinder blocks, and heavy sheet-iron boxes filled with sand.

In addition to mechanical vibrations, air currents and temperature changes affected the position of the two screens, but these difficulties were reduced to a minimum by avoiding drafts and maintaining a uniform room temperature.

Every possible precaution was taken to prevent jarring the apparatus while observations were being made, so that a moderate degree of stability was realized throughout most of the tests. However, when light intensity or bridge voltage was increased beyond a certain point the photo-electric cell resistances decreased very rapidly, and the sensitivity of the bridge increased to such a point that extremely minute vibrations were noticeable; this unstable region marked the limit to which the work could be carried under existing laboratory conditions.

RESPONSE OF BRIDGE

In determining the most convenient and logical way to express the response of the bridge to angular displacements of the torsion rod the conditions under which the bridge was being operated had to be considered.

In the usual operation of Wheatstone's bridge the two ratio arm resistances are maintained constant, and any changes in unknown arm R_4 are determined from knowledge of R_3 in accordance with the relation $R_4 = \frac{R_2 R_3}{R_1}$. In the present case, however, the ratio arm resistances are continually changing as the torsion system is deflected, while R_4 is kept at some constant value. The change in R_3 , then, determines the variation of the bridge ratio, which in turn measures the angular displacement of the mirror. This change in R_3 depends on R_4 and the photo-electric cell resistances R_1 and R_2 .

It therefore is desirable to choose a quantity that will measure the variations in R_1 and R_2 and at the same time be independent of R_4 . A study of the bridge equations indicates that ratio $\frac{\Delta R_3}{R_3}$ will satisfy these requirements; the following analysis was made to verify this point:

Let R_1 , R_2 , and R_3 be the initial values of bridge resistances and $(R_1 + \Delta R_1)$, $(R_2 + \Delta R_2)$, and $(R_3 + \Delta R_3)$ the corresponding values of bridge resistances for a change in light intensity, Δx , produced by an angular deflection of the mirror equal to $\Delta\theta$. $R_4 = \text{constant}$. For the initial balance:

$$\frac{R_1}{R_2} = \frac{R_3}{R_4}$$

For the new balance:

$$\frac{R_1 + \Delta R_1}{R_2 - \Delta R_2} = \frac{R_3 + \Delta R_3}{R_4}$$

Then:

$$\Delta R_3 = R_4 \left(\frac{R_1 + \Delta R_1}{R_2 - \Delta R_2} \right) - R_3$$

Solving:

$$\frac{\Delta R_3}{R_3} = \frac{\frac{\Delta R_1}{R_1} + \frac{\Delta R_2}{R_2}}{1 - \frac{\Delta R_2}{R_2}}$$

For the changes in light intensity used $\frac{\Delta R_2}{R_2}$ is very small compared to unity, and it may be neglected.

Therefore:

$$\frac{\Delta R_3}{R_3} = \frac{\Delta R_1}{R_1} + \frac{\Delta R_2}{R_2}$$

Thus the ratio $\frac{\Delta R_3}{R_3}$ is independent of R_4 , but it depends on the relative shapes of the two photo-electric cell characteristics. It is apparent that reasonably steep resistance-light intensity characteristics of the cells are desirable since both ΔR_1 and ΔR_2 are functions of the change in light intensity. However, if gas cells are used and they are operated near the point at which secondary ionization or flashover occurs the bridge balance becomes too unsteady to give results of any real value.

The constancy of $\frac{\Delta R_3}{R_3}$ with varying values of R_4 was checked experimentally, and the results are given in Table 2.

TABLE 2.—Constancy of $\frac{\Delta R_3}{R_3}$ with changes in R_4

Lamp voltage=70 volts
Lamp intensity=255 C. P.
Bridge voltage=45 volts
Angular displacement= 23.4×10^{-7} radians

R_4	R_3	$R_3 + \Delta R_3$	ΔR_3	$\frac{\Delta R_3}{R_3} \times 10^3$	R_4	R_3	$R_3 + \Delta R_3$	ΔR_3	$\frac{\Delta R_3}{R_3} \times 10^3$
1,000	560	470	90	16.10	11,000	6,300	5,300	1,000	15.90
2,000	1,270	1,062	208	16.35	15,000	8,210	6,890	1,320	16.10
6,000	3,610	3,040	570	15.80	20,000	10,630	8,880	1,750	16.45

The agreement is within the limits of the experimental error as determined by the accuracy with which the position of the interference fringes may be estimated.

CALIBRATION OF BRIDGE

Although the method developed in this investigation was to be used as a detector it is well suited for making absolute measurements of small angular displacements. In any case, it was desirable to calibrate the apparatus, determining its sensitivity and performance under varying voltages of bridge and intensities of light.

The method of calibration follows. A small angular deflection of the lever arm was produced by placing weights in a pan attached to a string running to the lever arm over a pulley (see fig. 18, *A*). The angle was then determined by observing the shift of the interference fringes produced by the change in the distance between the plates of the Fabry-Perot interferometer and substituting this quantity in equation (51). By means of the telescope and cross hair these fringe shifts could be estimated to the nearest tenth of a fringe, and

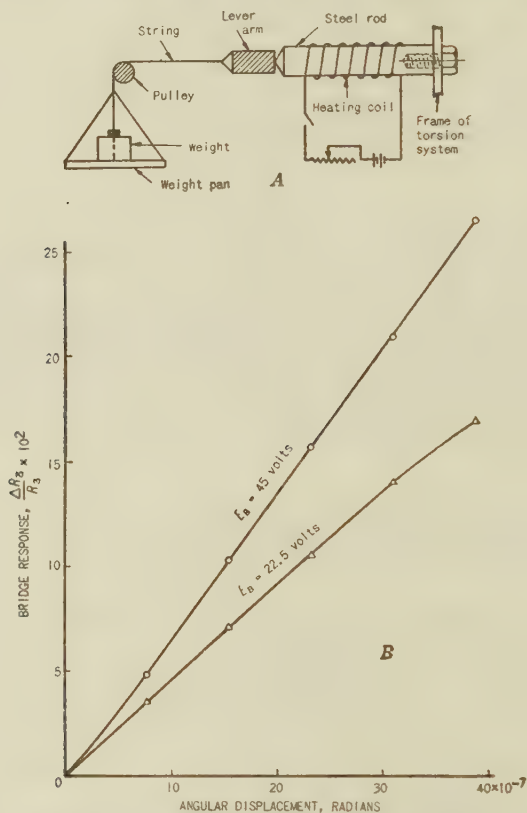


FIGURE 18.—*A*, Methods of producing angular displacements of torsion system; *B*, angular displacement-response characteristics. Lamp intensity, 255 candlepower; $R_4 = 10,000$ ohms; E_B = bridge voltage

this factor determined the limiting experimental accuracy of the calibration.

To be sure that the entire frame was not being twisted by the force on the lever arm, thus giving rise to erroneous displacements of the mirror, an alternative scheme of deflecting the torsion rod was used. One end of a bar of steel was bolted to the frame, and the other end was allowed to rest against the lever arm (fig. 18, *A*). Then by placing a small heating coil on this bar its temperature was raised; the resulting linear expansion caused the torsion rod to be deflected through a small angle. No external forces were brought to bear on the frame. Several points were checked thus, and results

obtained in the two cases agreed satisfactorily. This scheme demonstrated that the torsion frame was not subjected to any appreciable twist when the method of weights was used to displace the lever arm.

DISPLACEMENT-RESPONSE CHARACTERISTICS

The first apparatus characteristic determined was the relation between angular displacement of the torsion rod and the change in the bridge balance as indicated by the quantity $\frac{\Delta R_3}{R_3}$. The mean observations of numerous experimental runs are brought together in Figure 18, *B*.

It was found that for bridge voltages of 22.5 and 45 the balance could be determined to an accuracy of about 10 ohms. At 45 volts the apparatus could detect angular displacements of the order of

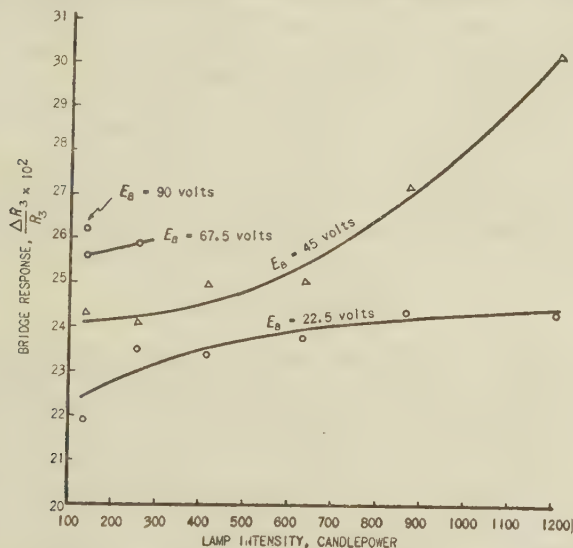


FIGURE 19.—Effects of intensity of lamp and voltage of bridge on response. Angular displacement, 39×10^{-7} radians; E_B = bridge voltage

2×10^{-8} radian, whereas at 22.5 volts this sensitivity was diminished to about 3×10^{-8} radian.

When used actually to measure small angles these results indicate that the method can determine displacements of 1×10^{-7} radian with reasonable accuracy under favorable conditions of mechanical stability.

EFFECTS OF INTENSITY OF LIGHT AND VOLTAGE OF BRIDGE ON RESPONSE

The investigation of the performance of the apparatus was concluded with a study of the effects of variations of bridge voltage and light intensity on the response. These observations are given in Figure 19. The curves indicate a rather sharp rise in response as intensity of the lamp increased for a bridge voltage of 45, whereas with 22.5 volts increases in the lamp voltage affected the response only slightly. In both instances, however, the response did not increase greatly even though the intensity of light was made

ten times its original value. The same is true of increases in voltage of bridge. The only advantage gained by either method was the added precision with which balance of the bridge could be obtained. On the other hand, as higher values of lamp or bridge voltage were applied the point at which the photo-electric cells became sufficiently ionized to cause flashovers was approached, and operating conditions became so unstable that accurate readings could not be obtained. This feature limited the magnitude of both intensity of the lamp and voltage of the bridge.

DISCUSSION OF EXPERIMENTAL WORK

The foregoing experiments emphasized the following facts:

1. The method could detect angular displacements of the order of 2×10^{-8} radian and could be used to measure angles as small as 10^{-7} radian with reasonable accuracy under favorable conditions of mechanical stability and at constant temperatures. The importance of these latter conditions can not be overemphasized.

2. The ratio $\frac{\Delta R_3}{R_3}$, which gave a measure of the angle of displacement of the torsion rod, was shown to be independent of the value of resistance R_4 used in the bridge circuit. On the other hand, it was a function of the shapes of the resistance-light intensity characteristics of the two photo-electric cells, as shown by the relation:

$$\frac{\Delta R_3}{R_3} = \frac{\Delta R_1}{R_1} + \frac{\Delta R_2}{R_2}.$$

3. The response of the apparatus was increased by an increase in either voltage of the bridge or intensity of the light. The advantage gained, however, was by no means commensurate with the increases in these quantities. The bridge balance could be determined more accurately, but this condition is limited to the region below which the cells become unstable owing to secondary ionization. Beyond this point increases in either voltage of the bridge or intensity of the light were accompanied by such unstable operating conditions that consistent results could not be obtained.

4. If further increase in the sensitivity of the apparatus is desired, the most likely methods of accomplishing this would be to lengthen the path of light from mirror to screens and to use finer screens. Both projects have limited application, for as the screens are made finer it becomes increasingly difficult to obtain a mirror of sufficient optical perfection to project an undistorted image of the sending grid on the receiving screen. In addition, the screens must not be made so fine that any appreciable diffraction occurs for wave lengths to which the photo-electric cells are sensitive.

APPLICATION OF METHOD TO REDUCTION IN BALANCE PERIOD

The advantages of incorporating into the torsion balance this system of measuring or detecting angular deflections are well illustrated in Table 3. The balances listed in Table 1 are given with their present sensitivities and periods. If each balance were operated as a null instrument and equipped with a suspension of such rigidity

that the gradient sensitivity was reduced to 1×10^{-8} radian, the periods would be reduced to the values shown in the last column.

TABLE 3.—*Decrease in torsion-balance periods*

[Approximate values]

Balance	Present gradient sensitivity, seconds of arc	Present period, seconds	Period for a gradient sensitivity of 10^{-8} radian, seconds
1. Eötvös single balance, 1890.....	36	1,355	10.3
2. Eötvös single balance, 1898.....	15	1,345	15.7
3. Eötvös double balance (small), 1925.....	16	1,405	16.0
4. Eötvös balance (miniature), 1908.....	6.4	875	15.7
5. Schweydar-Askania large balance.....	16	1,460	16.6
6. Schweydar-Askania Z-bar balance.....	8.8	1,275	19.5
	18	1,825	19.7
	11	1,440	19.9
7. Hecker, small model, 1925.....	9.7	1,085	15.8
	22	1,640	15.8
	12	1,210	15.9
8. Hecker, large model, 1927.....	23	1,770	16.8
9. Haff, 1928.....	11	1,400	19.2
	3.7	810	19.1
10. Haff, 1929.....	18	1,540	16.5
11. Oertling.....	9	1,315	19.9

Thus, with such an arrangement it will be possible to construct balances with periods of 20 seconds or less.

SEISMIC EFFECTS

As the sensitivity of the detector system is increased a point is reached at which the torsion-balance system becomes a seismometer, responding to the microseismic vibrations always present on the earth's surface. The average frequency of these vibrations is about 20 cycles per second. Their average displacement is zero, so that by using a galvanometer having a relatively long period in comparison with this frequency the effect of small earth tremors is rendered nil.

CONCLUSIONS

Torsion balances in their present form show periods of oscillation lasting from 14 to more than 30 minutes. Obtaining observations therefore requires considerable time, and temperature changes prove troublesome. By making the torsion balance a zero instrument by the addition of a fine restoring fiber and adapting the optical method of detecting small angular deflections described in this report balances can be produced with periods of less than 30 seconds without lowering the present standards of sensitivity.

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APPENDIX A

DERIVATION OF TORSION-BALANCE EQUATION

ANALYTICAL METHOD

The simplified equation for the potential at any point in a gravity field was found to be:

$$V = V_0 + gz + \frac{1}{2} (V_{xx}x^2 + V_{yy}y^2 + V_{zz}z^2 + 2V_{xy}xy + 2V_{yz}yz + 2V_{xz}xz). \quad (1)$$

V_0 = potential at the reference point.

It has also been demonstrated that only the horizontal components of gravity contribute to the deflection of the balance system. Denoting these components by the symbols X and Y :

$$X = \frac{\partial V}{\partial x}, \quad Y = \frac{\partial V}{\partial y}. \quad (2)$$

Combining equations (1) and (2):

$$\begin{aligned} X &= V_{xx}x + V_{xy}y + V_{xz}z, \\ Y &= V_{yx}x + V_{yy}y + V_{yz}z. \end{aligned} \quad (3)$$

These forces acting on the balance system produce a torque about the axis of the torsion wire. Denoting the turning moment by symbol F :

$$F = \int (Yx - Xy) dm. \quad (4)$$

The integration is taken over all masses of the suspended system of the balance, as all are acted on by the forces X and Y .

Replacing X and Y by their equivalents given in equation (3) and using the relation that $V_{xy} = V_{yx}$:

$$F = \int [(V_{yy} - V_{zz}) xy + V_{xy}(x^2 - y^2) + V_{yz}xz - V_{xz}yz] dm. \quad (5)$$

This equation applies when the balance beam and torsion wire coincide with the X and Z axes of the coordinate system, respectively. As the balance is rotated about the axis of the torsion wire the beam makes an angle, α , with the X axis, and equation (5) must be transformed to a new system of coordinates in which the new X axis again coincides with the balance beam. Let ξ , η , and ζ = the new coordinates and $\bar{X}$, $\bar{Y}$, and $\bar{Z}$ the new axes.

Old coordinates x , y , and z are related to the new ξ , η , and ζ by the relations:

$$\begin{aligned} x &= \xi \cos \alpha - \eta \sin \alpha, \\ y &= \xi \sin \alpha + \eta \cos \alpha, \\ z &= \zeta. \end{aligned} \quad (6)$$

Using (6) to transform (5) to the new set of axes:

$$F = (V_{yy} - V_{zz}) \left[\frac{\sin 2\alpha}{2} \int (\xi^2 - \eta^2) dm + \cos 2\alpha \int \xi \eta dm \right] \\ + V_{xy} \cos 2\alpha \int (\xi^2 - \eta^2) dm + (V_{yz} \cos \alpha - V_{xz} \sin \alpha) \int \xi \zeta dm \\ - (V_{yz} \sin \alpha + V_{xz} \cos \alpha) \int \eta \zeta dm. \quad (7)$$

The integrals appearing in equation (7) must now be evaluated. To aid this evaluation use is made of the fact that the balance beam and all other masses in the suspended system lie in a vertical plane through the $\bar{X}$ axis. Then $\eta = 0$ for all parts of the suspended system. Consequently, all integrals in which η appears as a common factor are equal to zero. Thus,

$$\int \xi \eta dm = 0, \\ \int \eta \zeta dm = 0,$$

and

$$\int (\xi^2 - \eta^2) dm = \int \xi^2 dm.$$

From this equation (7) reduces to the form:

$$F = \left[(V_{yy} - V_{zz}) \frac{\sin 2\alpha}{2} + V_{xy} \cos 2\alpha \right] \int \xi^2 dm + (V_{yz} \cos \alpha - V_{xz} \sin \alpha) \int \xi \zeta dm. \quad (8)$$

Quantity $\int \xi^2 dm$ is the moment of inertia of the suspended system about the axis of the torsion wire. Let it be denoted by K .

For all points on the balance beam ζ is zero. Hence, the integral $\int \xi \zeta dm$ is zero for all masses except the lower weight and its suspension. For the lower weight:

$$\int \xi \zeta dm = mbh.$$

Hence, equation (8) becomes:

$$F = \frac{K}{2} (V_{yy} - V_{zz}) \sin 2\alpha + K V_{xy} \cos 2\alpha + mbh (V_{yz} \cos \alpha - V_{xz} \sin \alpha). \quad (9)$$

At equilibrium moment F is opposed by torque T of the torsion wire.

$$T = \tau (\theta - \theta_0), \quad (10)$$

where τ = rigidity of wire and $(\theta - \theta_0)$ = net angular deflection.

Equating (9) and (10) and solving for $(\theta - \theta_0)$, the torsion-balance equation results. Thus,

$$\theta - \theta_0 = \frac{K}{2\tau} (V_{yy} - V_{zz}) \sin 2\alpha + \frac{K}{\tau} V_{xy} \cos 2\alpha + \frac{mbh}{\tau} (V_{yz} \cos \alpha - V_{xz} \sin \alpha). \quad (11)$$

APPENDIX B

The following discussion is presented to illustrate the significance of the second partial derivatives of the gravity potential to readers of limited mathematical experience.

Figure 20 shows two forces F_1 and F_2 acting at points 1 and 2 in space, respectively. As these forces are vector quantities they may be resolved into three components along the three coordinate axes. Thus, F_1^x , F_1^y , and F_1^z are the three components of F_1 , and F_2^x , F_2^y , and F_2^z are those of F_2 . Each of the components of F_1 will in general change in magnitude in moving from points 1 to 2. As an illustration, consider the change in F^x .

Instead of moving directly from 1 to 2, for purposes of calculation, the path is broken into its three coordinate parts, Δx , Δy , and Δz (see fig. 21). Now, in general the X component of force will change when moved in any of the three coordinate directions. Thus, a change will occur in going from points 1 to a in the X direction. Another change will occur between a and b in the Y direction and finally an additional change in going from b to 2 in the Z direction.

The second partial derivative of potential enables one to compute these changes in the various components of a force when moving from one point to another in space.

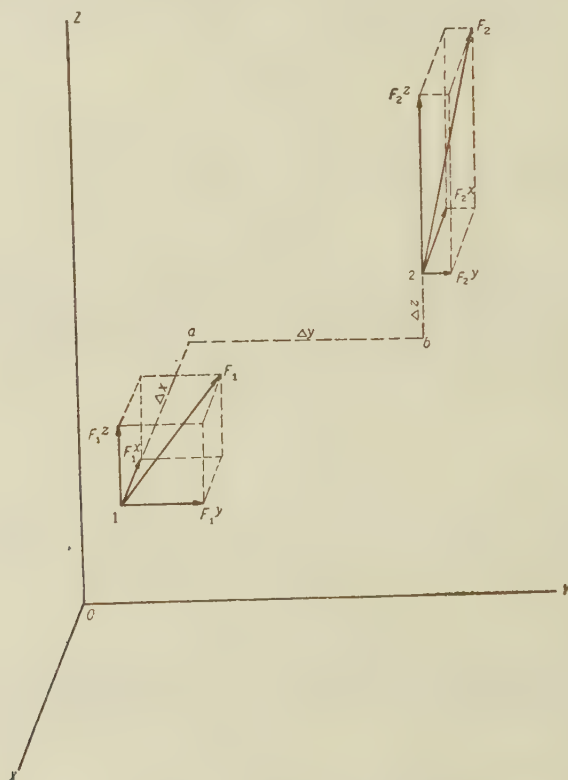


FIGURE 20.—Variation of a force in space

In this discussion, for simplicity only, the case in which the second partial derivatives are constants will be considered.

One interpretation of the derivative of a quantity with respect to a variable is that it expresses the rate of change of the quantity as the variable is changed at uniform rate. Thus, in mechanics the derivative of distance with respect to time gives the time rate of change of distance, in other words, velocity. Similarly, the derivative of a force with respect to direction expresses the space rate of change of the force in that direction, that is, the gradient of the force in the given direction.

For every point in space there exists a definite gravitational force; in other words, the force of gravity is a function of the three space coordinates. In the Cartesian system these coordinates are denoted by

$x, y,$ and z . Now, by definition the partial derivative of a quantity with respect to a particular variable is the derivative, considering all other variables constant. Hence, the partial derivative of the X component of gravity with respect to x gives the space rate of change of that component in the X direction, that is, the gradient of F^x in the X direction. Expressed mathematically:

$$(\text{Gradient } F^x)_x = \frac{\partial F^x}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial V}{\partial x} \right) = \frac{\partial^2 V}{\partial x^2}.$$

Similarly,

$$(\text{Gradient } F^x)_y = \frac{\partial F^x}{\partial y} = \frac{\partial}{\partial y} \left(\frac{\partial V}{\partial x} \right) = \frac{\partial^2 V}{\partial x \partial y},$$

$$(\text{Gradient } F^x)_z = \frac{\partial F^x}{\partial z} = \frac{\partial}{\partial z} \left(\frac{\partial V}{\partial x} \right) = \frac{\partial^2 V}{\partial x \partial z}.$$

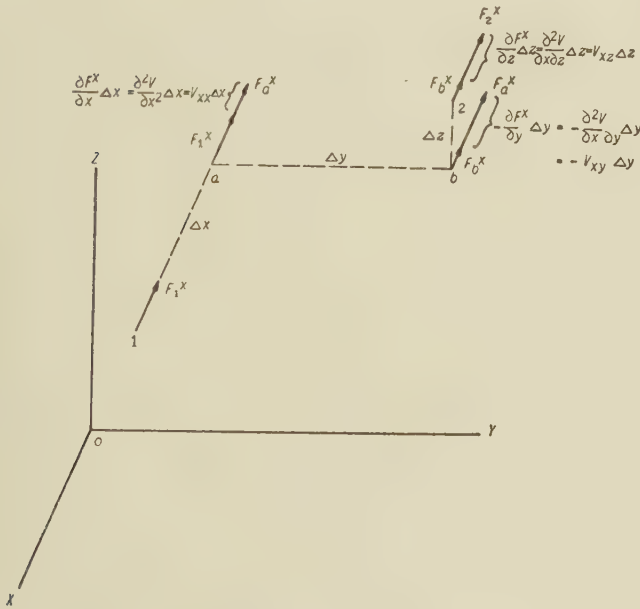


FIGURE 21.—Variation of one component of a force in space

As restriction of a constant gradient has been imposed these quantities remain constant within the balance system, and to find the total change in force simply multiply the gradient by the distance traversed.

Figure 21 illustrates the various changes in the X component of force in moving from point 1 to point 2.

From 1 to a :

$$\text{Gradient} = \frac{\partial F^x}{\partial x} = \frac{\partial^2 V}{\partial x^2} = V_{xx},$$

$$\text{Distance traversed} = \Delta x,$$

$$\text{Change in } F^x = V_{xx} \Delta x.$$

Therefore,

$$F_a^x = F_1^x + V_{xx} \Delta x.$$

From a to b :

$$\text{Gradient} = \frac{\partial F^x}{\partial y} = \frac{\partial^2 V}{\partial x \partial y} = V_{xy},$$

$$\text{Distance traversed} = \Delta y,$$

$$\text{Change in } F^x = V_{xy} \Delta y.$$

Therefore,

$$F_b^x = F_1^x + V_{xx} \Delta x + V_{xy} \Delta y.$$

From b to $\mathcal{Q}$:

$$\text{Gradient} = \frac{\partial F^x}{\partial z} = \frac{\partial^2 V}{\partial x \partial z} = V_{xz},$$

$$\text{Distance traversed} = \Delta z,$$

$$\text{Change in } F^x = V_{xz} \Delta z.$$

Therefore,

$$F_2^x = F_1^x + V_{xx} \Delta x + V_{xy} \Delta y + V_{xz} \Delta z.$$

Similar reasoning gives the changes in the other two components.

Thus

$$F_2^y = F_1^y + V_{yx} \Delta x + V_{yy} \Delta y + V_{yz} \Delta z,$$

$$F_2^z = F_1^z + V_{zx} \Delta x + V_{zy} \Delta y + V_{zz} \Delta z.$$



